Mathematical Patterns

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Introduction

Mathematics is a scientific field which has implementations in science, technology, in all other sciences and in the real world, and most of the time it is called as a science of patterns; it is not just the numbers and processes done with them. Mathematics is a tool which is used in understanding environment, solving problems, being creative and expressing things that cannot be seen otherwise. In a sunflower, in a leopard speckle, in the flow of water, in a golf course, in the rolling of a dice, in the living time of a star, in the shape of the world, in the pace of a runner, in weather forecasting, in a computer's processing, there is always a pattern. It can also be said that, in daily life, there is always mathematical relations in nature (Devlin, 1998). Understanding these relations and finding rules between these relations enters the field of algebra, which is one of the sub-branches of mathematics.

NCTM (2000) suggests that it is important to develop a different educational approach in algebra teaching known as a difficult area for children. Traditionally, algebra is introduced to the students at the end of primary education and in the early secondary education. Due to the abstract nature of algebra (Orton and Orton, 1999), this new information is seen as a completely compelling mathematical field by the students (Willoughby, 1997). As a result, it is seen that the thoughts on algebra education are changed. Researchers and Educators suggest that the natural development in algebra should be presented more gradually (Fouche, 1997, Willoughby, 1997). To do this, they state that it is needed to include algebraic concepts in primary education programs (Yackel, 1997; Orton and Orton, 1999).

Researchs supports NCTM striving for algebra education in an earlier period. Schliemann, Goodrow and Lara-Roth (2001) indicate that an education in which algebraic concepts presented in a concrete form first, then gradually the more abstract concepts chosen and later abstract representations are given can be better understood by primary students. Yackel (1997) describes that it is not meaningful to present quadratic formulas by abstract concepts to the primary school students when introducing algebra. Algebra includes reasoning and thinking strategies that help modelling, find patterns, and work with structures rather than just solving linear equations. Students need more than just thinking numerically. They also need to explain the ways to use the relationships and symbols between numbers (Yackel, 1997). Fouche (1997) states that to make algebra more beneficial, how to generalize the rules must be taught rather than memorizing them. Therefore, pattern and generalization concepts are seen as one of the most important issues of algebra.

Patterns, relations between patterns and generalization of patterns constitute the basis of algebraic thinking. In other words, students' ability to think algebraically is due to the recognition of the patterns, continuing of patterns and generalization (Smith, 2003; as cited in Steele, 2005, p. 142). Zaskis and Liljedahl (2002, p. 379), in mathematics, especially in the algebra, indicate that all concepts are related to the patterns and generalization of patterns. For this reason, many educators and researchers say that patterns and generalization of patterns are the heart and essence of mathematics.

Many philosophers, mathematicians and mathematical educators believe that patterns are a very important concept in learning and teaching of mathematics (Davis, 1984; Mottershead, 1985; National Council of Teachers of Mathematics, 2000). NCTM, while stating the program and evaluation standards of school mathematics, states that there are patterns all over the world, and therefore, the mathematical curriculums should help students deal with the mathematical models or definitions of these patterns (NCTM, 1989).

Both patterns and generalization concepts are considered the basis of mathematics. Therefore, these concepts are also very important for mathematics education. The importance of generalizing about the pattern is particularly emphasized. Because the structure of mathematics can be observed by means of searching patterns and relationships. These patterns and relationships become understandable, usable and expressible by generalizations about them. Thus, the patterns that can be expressed or sustained are gaining an important value for the passage to formal algebra (NCTM, 2000).

Schoenfeld and Arcavi (1988, p. 426) expressed that students' observing patterns and expressing them verbally will help the transition from arithmetic to algebra. According to them, the children who are engaged in the patterns begin to seek a regularity and relationship, and the generalizations they found encourage them in algebra. For Jones (1993, s.27), generalization is the heart of algebra and seeking patterns is an important step towards generalization.

The current studies in the context of early algebra education are investigating students' ability of generalization and whether students can find the rules in growing numerical patterns such as linear generalization problems (Stacey, 1989; Looney, 2004, Tanışlı, 2008). It is also stated that these subjects will constitute a strong basis for subsequent algebra learning (Kenney and Silver, 1997, Orton and Orton, 1999; Lannin, 2003; Schliemann, Caraher, Brizuela, Earnest, Goodrow, Lara-Roth and Peled, 2003; Ferrini -Mundy, Lappan and Philips, 1997). Schultz (1991) claim that early student's ability to develop more formal mathematical concepts should be based on their informal understanding of the pattern. Ferrini-Mundy et al. (1997) suggest that pattern activities can help students show relations, think about these relations and identify these relations. "Patterns, Relations and Functions" is located as one of the NCTM's algebra standards (2000). NCTM (2000) suggests that in order for students to generalize geometric and numerical patterns in a more complex manner, in order to show patterns and functions verbally, or as table and graphics; It recommends that students should participate in pattern activities from the young age.

As a result, the concepts of pattern and generalization are very important in mathematics education. In particular, in terms of the basis of algebra education, they must have experiences with the pattern and generalization at an early age and learn these issues. Students who have learned patterns and generalization at an early age will not have problems when they encounter abstract symbols, notations, and rules in algebra education. Based on here, it is important to define what pattern and generalization means; since they have a very important place in mathematics, especially in the algebra.

Pattern and Generalization Concept: Current Use and Mathematical Language

The concept of pattern is not found in the old math programs applied in Turkey. This concept is used for the first time in the renewed math program in 2004. Therefore, the "pattern" is not a very well-known and used word in the daily language. The concepts of "pattern" and "generalization" are described in the Turkish Language Association Dictionary and the Oxford Dictionary:

> *Pattern: The development of events or objects in an orderly way following each other. For example, the days of the week constitute a pattern (www.tdk.gov.tr).*
>
> *Pattern: (1) The regular way in which something happens or is done.*
>
> *(2) Pattern for something an excellent example to copy.*
>
> *(3) A regular arrangement of lines, shapes, colours, etc. for example as a design on material, carpets, etc.*
>
> *(4) A design, set of instructions or shape to cut around that you use in order to make something.*
>
> *(5) A small piece of material, paper, etc. that helps you choose the design of something (www.oxfordlearnersdictionaries.com).*
>
> *Generalization: To collect, circulate, determine similarity relations between assets or events in one thought (www.tdk.gov.tr).*
>
> *Generalization: A general statement that is based on only a few facts or examples; the act of making such statements (www.oxfordlearnersdictionaries.com).*

When the "pattern" and "Generalization" concepts are referred to by mathematically, a series of patterns, which can be modeled by a mathematical function, orderly sequenced, an object, a shape or numbers are called as pattern, and finding the mathematical function in this sequence of numbers as a generalization. There are also many definitions of mathematicians and mathematics educators related to the pattern.

> *Regularly lined, poem formed by recurring objects or shapes (Olkun and Toluk* Uçar, 2009, p. *120).*

> *A numerical or spatial regularity (Papic and Mulligan, 2005, p. 609).*

As mentioned earlier, the pattern is in all areas of life. Due to these patterns in daily life and the environment, children are naturally predisposed to generalize. Deloache, Miller and Pierroutsakos (1998) indicates that children try to use patterns and generalization.

Historical Development of Pattern Concept

Although there is no information on the historical development of the pattern concept, it is stated that the concept of patterns originated with the emergence of the number theory (Ore, 1948; as cited in Beougher, 1971). It is said that the most important of the groups related to the emergence of number theory is a group consisting of ancient Greek mathematicians known as Pythagoreans. This group consists of students who benefit from Pythagoras' teachings in a school, established by Pythagoras who gave his name to a famous formula in geometry (550 BC). Number patterns are also seen as one of the most important parts of the number theory. Some of these patterns were found by Pythagoreans and they provided names such as friendly-amicable numbers, excessive numbers, deficient numbers, and perfect numbers. For example, divisors of the numbers 1184 and 1210, excluding 1184 and 1210, are as follows.

The sum of the divisors of 1184 from these numbers 1210, and the sum of the divisors of 1210 gives the number of 1184. Based on here, the Pythagoreans have made the name of this type friendly-amicable numbers.

Likewise, the divisors of 6, 8 and 12 others than themselves are seen in Table 1.

Table 1. Examples of Perfect, Excessive and Deficient Numbers

	Number	Divisors	Sum of Divisors
Perfect Numbers	28	1, 2, 4, 7, 14	1+2+4+7+14 = 28 (28 = 28)
Excessive Numbers	12	1, 2, 3, 4, 6	1+2+3+4+6 = 16 (12 < 16)
Deficient Numbers	15	1, 3, 5	1+3+5 = 9 (15 > 9)

According to Table 1, the number 28 is obtained when the divisors of 28 added. These types of numbers were called as the perfect numbers by Pythagoreans. Even because the first perfect number is 6, they claimed that God created the world in 6 days. Similarly, since the sum of the divisors of the number 12 is greater than 12, these types of numbers were called excessive numbers, and as the sum of the divisors of the number 15 is less than 15, this type of numbers were called deficient numbers.

In addition to such patterns, the Pythagoreans have shown the numbers with points and the geometric configurations of those numbers.

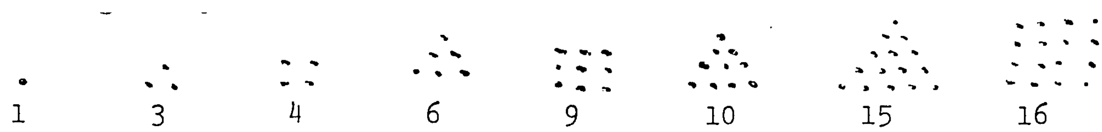


Figure 1. Displays of Triangular and Quadrants

As seen in Figure 1, the Pythagoreans also showed patterns via shapes. Numbers such as 1, 4, 9, 16 were called quadrant numbers; and numbers such as 1, 3, 6, 10, 15 were called as triangular numbers. After that time, many new patterns were uncovered with the development of algebra. Eratestones (230 BC) revealed prime numbers, Fibonacci revealed a pattern named after him, and Gauss revealed a pattern related to the total of sequential numbers (Ore, 1948; as cited in Beougher, 1971).

Types of Patterns

Primary-age students have two types of patterns to discover. One of them is repeating patterns and the other is the growing patterns (Warren and Cooper, 2006). Mostly these patterns are used in finding generalizations within their elements. In this type of patterns, answer is sought for questions such as "What comes next? What is the repeating part? What are the missing elements? " These primary level activities are considered as finding the pattern in a data set that mostly shows a single change (Blanton and Kaput, 2004). Repeating and growing patterns lead to the functional thinking of students in early ages and help them better understand the relationships between the two data sets.

Repeating Patterns

Recurrent patterns are the continuous self-repetitions of a group of elements. These types of patterns have a repeat unit (Papic, 2007). Liljedahl (2004) states that it has a circular structure in which a small part of the repeating patterns can be generalized by repeating a small unit of a pattern again and again. He also stated that this small unit may change due to features like size, shape, magnitude, and direction, and called this part as pattern repeat unit, element, portion or part.

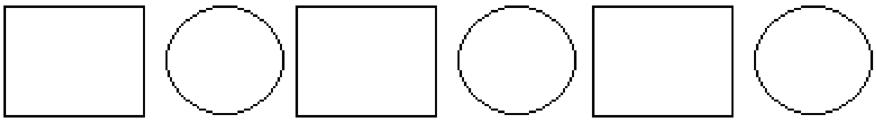
For example;

∇ ∇ ∇ ∇… pattern where ∇ is the repeat unit

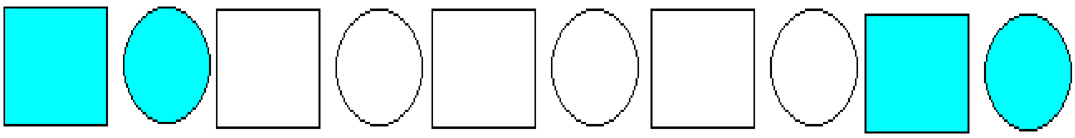
←↑↓→←↑↓→←↑↓→… pattern where ←↑↓→ is the repeat unit etc.

Various impressions are used in such patterns in different forms. For example ∇ ∇ ∇ pattern with geometric shapes, can be represented with movements such as "Sit down, get up, sit, get up, sit, get up", with sounds such as "drum tone, ringtone, drum tone, ringtone, drum tone, ringtone"; with letters such as "ababab" or with feelings such as "smooth, rough, smooth, rough, smooth, rough, smooth, rough". There is a sequence that students will follow when looking for a repeating pattern:

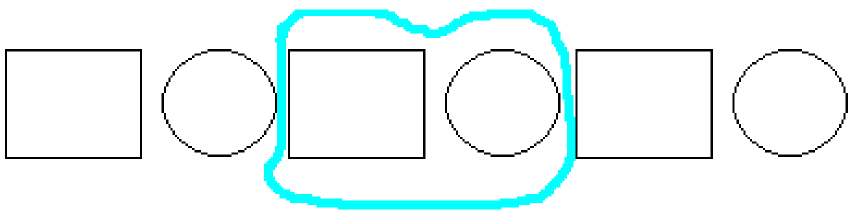
1. Copying the pattern: In the first phase students can be given a pattern as follows and be asked to copy this pattern.



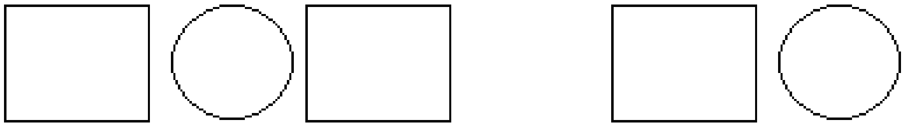
2. Continuing the pattern: In this stage, the following type of questions can be asked to ensure that children are realizing the repeating pattern. "What shape comes after the hoop? Which shape comes before the quadrangle? ". Then, the students may be asked to extend the pattern on both sides.



3. Identifying the repeating element: At this stage, the pattern can be uttered loudly (quadrangle, circle, quadrangle, circle, quadrangle, circle) and repeat unit can be asked to students. Then they may also be asked to encircle the repeat unit with a piece of rope.



4. Pattern Completion: A repeating pattern can be created, and children can be asked to continue the pattern and determine the repeating element. Here, some elements of the formed pattern can be removed by wanting children to close their eyes. When children turn their eyes again, they can be asked what element is removed.



5. Creating a pattern: At this stage, children may be asked to form their repeating patterns. After they have created them “Why is this a repeating pattern? Will you show me the repeating part? How does this pattern continue?” etc. questions can be asked.

6. Transferring a pattern to a different environment: This process is very important at all levels of mathematics. This stage can help children in small age developing their common and different aspects in developing different representations**.** These differences are generally superficial differences. For example, sit down, get up, sit down, get up instead of triangle, square, triangle, square use etc. The partnerships are related to the structuralism of the mathematics. For example, the pattern consists of two elements and an element is continuously follows the other etc. (Warren and Cooper, 2006).

Papic (2007) says that repeating patterns can be discussed in 3 parts, in the form of linear, cyclical and hopscotch. He states that repeating patterns are mostly presented in a linear form like a flat line, and that linear repeating patterns may extend forever in accordance with different instructions. Simple repetitions such as “Abababab” are one of the typical samples of linear repeating patterns and the elements of such patterns can be presented vertically, horizontally, or diagonally. The first or end point of cyclic repeating patterns are not exactly clear. Day-night, seasons or the limit of patterns generated at the edge of a polygonal region and simply repeats can be shown in this type of pattern (see Figure 2.a). Hopscotch patterns are used to investigate the children’s abilities of repeating units of squares to rotate horizontally and vertically. Such tasks are investigating how students see the changes in the direction of the pattern and their transformation related skills (see Figure 2.b) (Papic, 2007).

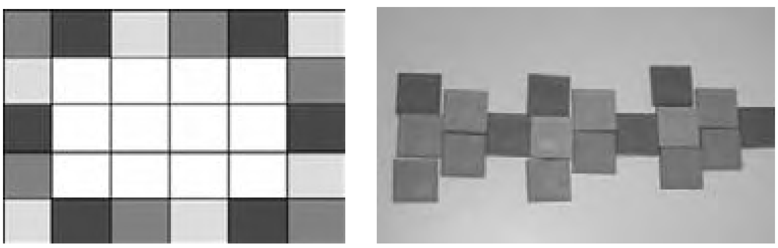


Figure 2. Cyclical and Hopscotch Repeating Pattern Examples

Growing (Changing) Patterns

Growing patterns are also known as numerically increasing or decreasing patterns. Growing patterns are the patterns that are followed by an expanding or shrinking course of relations between terms (Olkun and Yeşildere, 2007, p. 13). Each term in such patterns connected to each other with a rule. In such patterns, a generalization or an algebraic relationship is searched next to continuing the pattern (Tanışlı and Olkun, 2009, p. 11). In addition, it is stated that the growing patterns can be considered as a starting point in the concept of generalization, equation and function and algebraic thinking (Van De Walle, 2004). Growing patterns are grouped in four different ways (Tanışlı and Olkun, 2009, p. 11).

Arithmetically Growing Patterns: A follow-up of each term is obtained by adding a constant number is called an arithmetically growing pattern. The rule or relationship of the pattern may be explained by a linear equation. Therefore, it is also called linear growing patterns. An example of arithmetic growing ing patterns is given Figure 3.

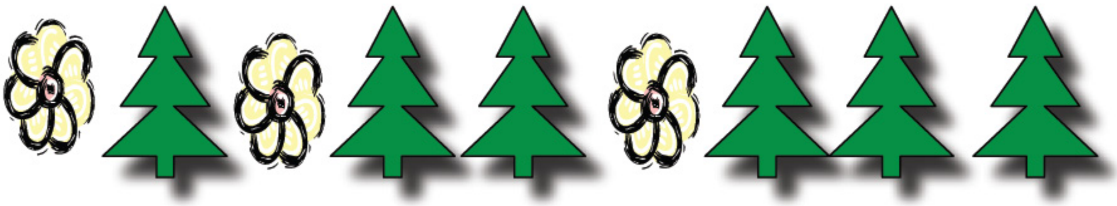


Figure 3. Arithmetically Growing Pattern Sample

The step of the pattern is called the range where the pattern shows similar change (Olkun and Yeşildere, 2007, p. 13). There is a flower and a tree in each step of the pattern in the sample. However, as the steps increase, an extra tree is added to each step. As given in the example, arithmetically growing patterns can be presented in shape, and they can also be presented in the form of numbers, table and verbal problem.

Geometrically Growing Patterns: The patterns in which successive pattern elements change within a ratio are called geometrically growing patterns. The rule or relationship in such patterns may be explained by the exponential equations. For example, for the n term of a pattern whose terms a_1, $a^1.r$, $a_1.r^2$, $a_1.r^3$, $a_1.r^4$, …; if the rate of the first term and the two consecutive terms are a_1, $a_n = a_1.r^{n-1}$ rule can be used. An example of geometrically growing patterns is given Figure 4.

Input	Output
1	2
2	6
3	18
4	54
5	162
…	…
n	2.3^{n-1}

Figure 4. Geometrically Growing Pattern Sample

The pattern in the sample is given outputs for each input. As the inputs progress, there is always a 3-times rate between the output numbers. For this reason, the elements of the pattern continue as 2, 2.3, 2.3^2, 2.3^3, 2.3^4, … This sample is presented in the pattern table format. In addition to this presentation, the geometrically growing patterns can be presented in the form of numbers, shape, and verbal problem.

Increasingly Growing Patterns: Patterns in which the differences between the successive terms increase or decrease are called increasingly expanding patterns. There are no fixed differences between terms in such patterns. However, when discrepancies between the differences are viewed, fixed differences are achieved in step 2 or 3. In addition, the rules of such patterns can be explained by the second or third order equations. For example; providing a, b and c are fixed and a_n is the n term, the rules of such patterns can be expressed in the form of $a_n=a.n^2+b.n+c$ or $a_n=a.n^3+b.n^2+c.n+d$. The rule is also called quadratic growing patterns which can be expressed in a quadratic equation. A sample is given to growing patterns increased in Figure 5.

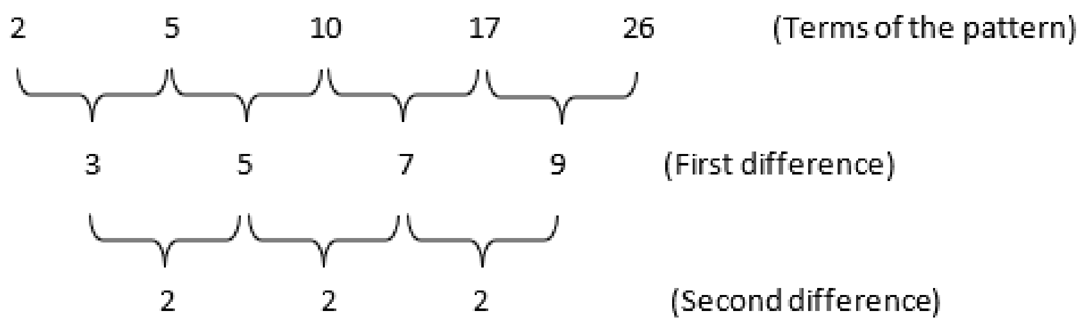


Figure 5. Growing Pattern Sample by Increasingly

The difference between the terms of the pattern in the sample is not a constant number. The differences between terms are 3, 5, 7, 9, ... However, it is seen that the difference between each difference is 2. This shows that the rule of the pattern can be expressed in a quadratic equation. In other words, this example of pattern is also a quadratic growing pattern. Provided that a_n is the n^{th} term, the rule of the pattern is in the form of $a_n=n^2+1$. This pattern is presented in the form of number sequence. In addition to this presentation form, growing patterns can be presented in the form of table, shape, and verbal problem.

Other Patterns: There are also patterns that do not belong to the class of arithmetic, geometric and increasingly expanding patterns, but whose terms change in a regularly. Fibonacci numbers, a very famous number sequence, and the Pascal triangle are examples of this type of pattern. A sample is given to other patterns in Figure 6.

A pair of rabbits (one male, one female) is put in the corral. Two months later, these rabbits have one male and one female offspring. After that, a couple will have offspring every month. This pattern continues: after two months each rabbit pair has a pair of offspring, and every month thereafter an additional pair of offspring. How many pairs there will be at the end of a year?

Figure 6. An Example of Other Type of Pattern

This pattern is called Fibonacci pattern. The terms of this pattern are obtained by adding the two previous numbers. This pattern does not correspond to an arithmetic pattern; it is also neither a geometric nor a growing pattern. The solution of the pattern presented above as a verbal problem is given in the table below (see Figure 7).

Ay	Tavsan Çiftleri	
1	1	
2	1	
3	2	After two months, the first couple has two offspring.
4	3	The first couple has two offspring for the second time.
5	5	The first offspring will have two offspring.
6	8	And so it continues.
7	13	And so it continues.
8	21	
9	34	
10	55	
11	89	
12	144	

Figure 7. Solution of Fibonacci Pattern

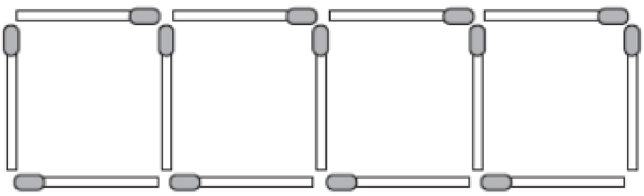
According to this, 144 rabbits pair will be formed at the end of 12 months. The terms of this pattern are in the form of 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... As in the case of such patterns, it can be presented in the form of verbal problem, in the form of a table, shape and number series (Olkun and Yeşildere, 2007, p. 13-20; p. 11-16). Therefore, these presentation forms are described, and the examples of these presentations are given in the next section.

Presentation Formats of Pattern

Types of patterns are examined in two main categories, repeating, and growing as described above. In addition, there are different forms of presentations to present any type of patterns. These presentation formats can be mentioned in different forms such as shape (visual), table or graphics, number sequences and verbal problems.

Patterns Presented in Shape (Visual) Format

The patterns presented in shape can be in the form of an increasing series of dots, in the form of shapes from matchsticks or toothpicks, shapes from unit squares or the patterns formed from blocks or tiles (Ley, 2005). In addition, shape patterns of real-world objects such as flowers, tree or geometric shapes can also be created (see Figure 3). The question in Figure 8 in TIMMS can be given as an example of the patterns presented in shape form.



13 matchsticks have been used in order to form the 4 squares in the figure. By using the same method how many squares can be formed by using 73 matchsticks?

Show your calculations used for finding the answer.

Answer: ....

Figure 8. A Pattern Question Presented in TIMMS

The patterns presented in concrete and visual form are seen easily than symbolically presented patterns. Bruner (1966), also, states that the orderly use of concrete, visual and symbolic stages in new learning will provide students with convenience. The aim of the shape patterns is to support students by thinking with visual approaches and to find an alternative way to numbers. Another goal of figure patterns is to diversify the ways of problem solving. It is also stated that the figure patterns can make students make changes when necessary and to help them create a new step from one step. It is also said that shape patterns are more fun than other forms of presentation for students (Orton, Orton and Roper, 1999, p. 122).

Figure patterns can be formed from the dots, can be formed from unit squares, and tiles. Figure 9 shows an example of a linear expanding pattern created with unit squares.

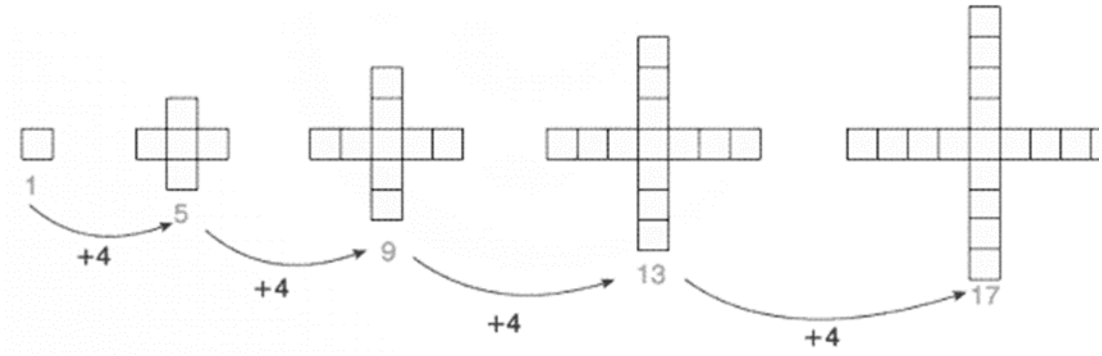


Figure 9. A Pattern Presented in the Form of Linear Expanding and Shape

This pattern has started with a square and is a shape pattern that has continued by adding a square to 4 sides on other steps. When finding the pattern, it is necessary to determine how many squares in each step first. Then, the differences between the number of square numbers in each step should be examined and whether the difference changes or not. The difference between the square numbers in each step in this pattern is constant and 4. Therefore, this pattern is a linear expanding pattern, and the rule of the pattern is a_n and n^{th} term is in the form of an = 4n-3.

Patterns Presented in Table or Graphics Format

Another form of presenting patterns is the presentation in the form of table. An example of this type is seen in Figure 10. In this pattern question, there is a table in which the value column B changed according to the value in column A. In such tables, students are required to find the relationship between the number in column A and the number in column B.

Çalının boyu (cm)	Gölge boyu (cm)
20	16
40	32
60	48
80	64

Yukarıdaki çizelge, farklı boylardaki dört çalının sabah saat 10:00'daki gölge boylarını göstermektedir. 50 cm boyundaki bir çalının sabah saat 10:00'daki gölge boyu nedir?

Ⓐ 36 cm
Ⓑ 38 cm
Ⓒ 40 cm
Ⓓ 42 cm

Bush height (cm)	Shadow length (cm)
20	16
40	32
60	48
80	64

The table shows shadow lengths of four bushes of differeing heights at 10.00 A.M. What is the shadow length of a 50 cm bush at 10.00 A.M.?

A) 36 cm
B) 38 cm
C) 40 cm
D) 42 cm

Figure 10. A Pattern Question Presented in TIMMS

It is stated that the table presentation form plays a critical role in a pattern search processes through the outputs that students can systematically record in each row (Schliemann, Carraher and Brizuella, 2001). On the other hand, researchers such as English and Warren (1998) state that especially in the understanding the concept of variable, it is very important to make generalizing from the data tables. In addition, some researchers also reveal the benefits of the table form presentation in the understanding of students in particular linear functions and seeing functional relations (Schliemann, Carraher and Brizuella, 2001; Martinez and Brizuella, 2006). It is also said that it can help students to understand the value of the table when defining a pattern (Pegg and Redden, 1990).

In the patterns presented in the table format, the first column consists of the inputs and the second column consists of outputs of these inputs. In Figure 11 a pattern which is presented in the form of table and expanding quadradically is given.

Input	Output
1	0
2	3
3	8
4	15
5	24
6	35
…	…
n	?

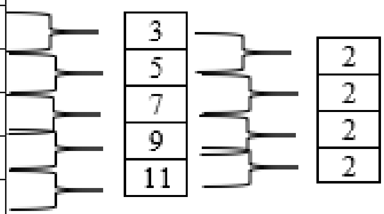


Figure 11. A Sample of Pattern Presented Table Format and Expanding Quadratically

In the above pattern, the number of inputs is 1, while the output number is 0, and while the number of inputs was 2, the number of outputs was 3 and the pattern continues this way. Looking at the differences between the output numbers, it seems that the differences are not constant, and they change. Therefore, it can be mentioned that this pattern is an increasingly expanding pattern. When we look at the difference of differences, it is seen that there is a constant difference of 2. Therefore, it can be said that this pattern is a pattern whose rule can be expressed by a quadratic equation. The rule of the pattern is $a_n=n^2-1$ provided that a_n is the n^{th} term.

The Patterns Presented in the Form of Numbers

Another form of presenting patterns is number sequences. These are also named as skip counting. Generally, patterns are presented in this form in mathematics textbooks. In this presentation format students are required to determine the relationships between the terms given and to write the numbers of which are not given. It is also expected students to express the rule of the pattern based on the relationship they found (Ley, 2005). The terms of these types of patterns can be presented from left to right by issuing a comma or with a certain amount of space. An example of this presentation can be given as follows.

8 14 20 26 32 … …

In this sequence, the first 5 terms are given, and students are required to find the two terms coming afterwards. Then, in order to find the n^{th} term, students are expected to find a rule. In this presentation format, it should be ensured that the space between the terms are equal in the patterns. Because the numbers arranged at different distance may confuse students. For example, when the terms of the Fibonacci sequenced as follows students will experience great problems at catching this pattern (Burke and Orton, 1999).

1, 1, 2, 3, 5, 8, 13, …

This kind of number sequence patterns may cause students to have problems in finding the relationships between terms. Therefore, the patterns of the number sequence are also presented as follows.

1. adım:	*8*				
2. adım:	*8*	*14*			
3. adım:	*8*	*14*	*20*		
4. adım:	*8*	*14*	*20*	*26*	
5. adım:	*8*	*14*	*20*	*26*	*32*

In this type of patterns, students can find the rule of the pattern by relating to the number of steps with the last number in that step. For example, in this pattern, the last number in step 1 is 8, the last number in the second step is 14, the last number in step 3 is 20, the last number in step 4 is 22 and the last number in step 5 is 32. In view of the number of steps and the last number, it is seen that the last number 6 more than 2 times of the number of steps. Based on here, the rule of this pattern is $a_n=2n+6$.

The Patterns Presented in the Form of Verbal Problem

The patterns may be presented in the form of verbal problems or stories. One of the best-known problems in this type is the problem of "handshake" (Burns, 2000; Van De Walle, 2004).

> *If 2 people in a group handshake each other, it counts 1 time. If you have 3 people in the group, they handshake 3 times among themselves. How many handshakes will there be if there are 4 people in the group? How many handshakes will you have if there are 6 people in the group?*

Another example of this presentation style is presented below (Figure 12).

You have a rope and we cut it in the form "U". We do it repetitively for the rest of the rope. Accordingly;

1. How many pieces are formed when the rope has been cut once, twice, and three times?

How many pieces are formed when the rope has been cut 4 and 8 times?

2. Could you find a rule between the cut times and number of pieces? Express your answer verbally.

3. Can you form a symbolic rule if we name number of cuts as "K" and number of pieces as "P"?

P=

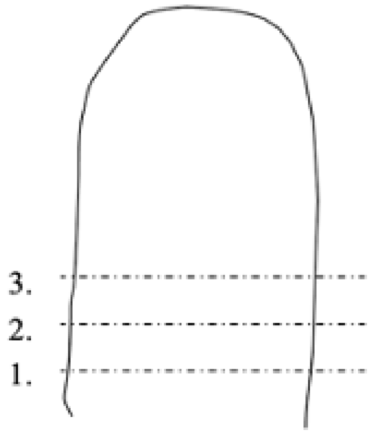


Figure 12. Number of Cuts Problem

Conclusion

NCTM recognized the importance of a different educational approach to teaching algebra. Because this field is very difficult for students. Traditionally, algebra is given to students in the 6-8. grades or high school ages. Despite the abstract nature of algebra, this sudden knowledge has been expressed as a complete struggle for students (Willoughby, 1997).

As a result, there has been a recent change in thinking about algebra education. Researchers and educators today suggest a more stepped and natural developmental education in algebra (Fouche, 1997; Willoughby, 1997). It is stated that such an education will only be possible if algebraic concepts are introduced gradually from the first years of primary education (Yackel, 1997; Orton & Orton, 1999).

Algebra education, which started in the 7th grade in Turkey, started in the first years of primary education with the Primary Education Mathematics Curriculum changed in 2004, and this process continued in the Mathematics Curriculum changes made in the following years. In particular, the subject of "Pattern and Tessellations" was placed in the first years of primary education, where it was aimed that students go to the algebra subjects in the future with a better foundation by generalizing and searching for relationships in patterns. In this context, besides what the concept of pattern is, the necessity of using pattern types and different presentation styles in lessons for different grade levels has emerged. Because in the studies conducted, it has been observed that students achieve different levels of success in different pattern types or different pattern presentation styles (Pegg & Redden, 1990; MacGregor & Stacey, 1993, 1995; English & Warren, 1998; Dobrynina, 2001; Lannin, 2003; Tsankova, 2003; Looney, 2004; Tanışlı, 2008; Yaman & Umay, 2013). In order to minimize these differences, it is necessary to ensure that students work with all pattern types and pattern presentation styles. Because the better the children are in patterns and relationships at this age, the more solidly they will move to algebra in the following years.

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