The Power of Inductive Reasoning in Mathematics Competitions

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Introduction

It is common to hear children as young as four years old utter the statement "all birds can fly". This statement might seem rather ordinary and insignificant at first. However on closer scrutiny, one would discover that this statement holds plenty of weight. Why is that so? Because for children to conclude that "all birds can fly", it seems to suggest that they have seen all species of birds and can therefore arrive at that remarkable conclusion. In reality however, it is extremely unlikely that children of that age have seen all species of birds available to make that conclusion, let alone experienced scientists who spend their lifetime studying birds. So how then are they able to arrive at that conclusion? What thinking processes are behind the statement that "all birds can fly"?

Inductive Reasoning

The ability to observe characteristics or properties about living things or inanimate objects in the surrounding natural environment and notice patterns is a natural inclination of the human mind (Mason, 1996). This inadvertently allows us to obtain scientific knowledge (Polya, 1945). It is second nature to observe that a few species of birds, such as crows or seagulls, can fly and naturally extend this reasoning to all other types of birds. Such is known as inductive reasoning, which is a process of analysing particular cases or observations and inferring to arrive at a general conclusion or rule (Polya, 1945). In other words, reasoning from particular instances is extended across all instances in order to reach a general conclusion. This is contingent upon detecting similarities across cases, which is espoused by Klauer (1988), and consists of looking for patterns and relationships within the given numerical data and figures (Neubert & Binko, 1992). According to Klauer (1988, 1992), inductive reasoning involves methodically and rigorously comparing objects with the ultimate objective of seeking out similarities and differences amongst the specific cases or relations. One can therefore say that inductive reasoning adds to the presently available information (Klauer, 1999) which is not an easy feat. In fact, inductive reasoning has even been considered as an ability to produce fresh perspectives, a productive characteristic of humans (De Koning et al., 2003; Sternberg & Gardner, 1983).

Importance of Inductive Reasoning

Since inductive reasoning is such a natural ability of the human mind, its importance to human society cannot be undermined. It is central to general human intelligence (Carroll, 1993) which underpins the performance in complex tasks from different subject content areas. The integral and foundational role that inductive reasoning plays in mathematics learning and problem solving (NCTM, 2000) is testament to its centrality and importance in human development. It is pervasively present and useful in many problem solving activities (Nisbett et al., 1983) and is therefore commonly used by many mathematics educators (Polya, 1954). In addition, inductive reasoning has been found to play an important role in developing expertise in a skill or content domain and has even been said to predict academic performance and test scores well (Pellegrino & Glaser, 1982).

Inductive reasoning, being such an important ability in mathematics, can therefore be said to be fundamental to problem solving involving higher order thinking (HOT) skills. Behind the acquisition of HOT skills lies the ability to recognize patterns and generalise, which constitutes inductive reasoning. This ability is closely associated with a person's knowledge of numbers and being able to identify relations within numbers (Haverty et al., 2000). Having "number sense", which is having an instinctive feel for numbers together with its uses, interpretations and relationships (NCTM, 1991; Howden, 1989), underpins all these. According to Greeno (1991), "number sense" includes capabilities such as being able to flexibly compute mentally, estimation of numerical data and evaluation. All these can be developed in stages through exploration of numbers in various contexts and drawing relationships between them without being subjected to the constraints of available algorithms or formulas (Howden, 1989).

The Singapore Mathematical Olympiad

A context where "number sense" is pushed to its limits is mathematics competitions. In such competitions, knowledge is extended beyond the confines of the school curriculum to help students inculcate an appreciation for mathematics and learn advanced mathematical thinking skills which would not have been accorded primary importance in the school curriculum (Toh, 2012). A mathematics competition that perhaps best embodies the extension of knowledge beyond the school curriculum is the Singapore Mathematical Olympiad (SMO), which is arguably the most eminent national mathematics competition in Singapore. Every year, the SMO garners the most number of participating students in Singapore and information is readily available from the Singapore Mathematical Society publications (Toh, 2012). According to official documents, the objectives of the SMO are two-fold; (1) to test the ingenuity and mathematical problem-solving ability of participants, and (2) to discover and encourage mathematical talents in Singapore schools (Chua, Hang, Tay & Teo, 2007).

Objectives of Paper

With the two objectives of the SMO in mind, it is not hard to see the importance of having good "number sense" to effectively reason inductively in problem solving contexts. Since inductive reasoning is so vital, how then does it feature in the SMO? How do the questions in the SMO enunciate the features of inductive reasoning? How can inductive reasoning be used as a cognitive strategy to tackle questions in the SMO? These are questions that would be of interest in this paper. Hence, the main aim of this paper is to illustrate the importance of inductive reasoning in the context of the SMO questions by showing that it is an important cognitive tool for solving SMO problems.

Inductive Reasoning in SMO Problems

To illustrate how inductive reasoning can be utilized in SMO problems, it is first necessary to reiterate what inductive reasoning is. Inductive reasoning, as seen from the above review of literature, is the process of analysing particular cases or observations and inferring to arrive at a general conclusion or rule. In the process of analysing particular cases, similarities, patterns and relationships are being drawn. The analysis of particular cases can be a very powerful vehicle or strategy for approaching problems such as those depicted in Figure 1.

Q1.

Among the five numbers $\frac{23}{44}, \frac{24}{45}, \frac{25}{46}, \frac{26}{47}$ and $\frac{27}{48}$, which one has the smallest value?

(A) $\frac{23}{44}$ (B) $\frac{24}{45}$ (C) $\frac{25}{46}$ (D) $\frac{26}{47}$ (E) $\frac{27}{48}$

(SMO 2017)

Q2.

The last two digits of 9^{2004} is

(A) 21 (B) 81 (C) 09 (D) 61 (E) 01

(SMO 2004)

Figure 1. Two SMO Questions of Inductive Reasoning Type 1.

As calculators are not allowed in SMO, it would be very difficult to tell which number has the smallest value without converting the five numbers into a decimal form in Q1. As such, another strategy is needed to determine which number is the smallest. Using the idea of analysing particular cases, a particular case can be conjured that mirrors the ordered sequence of five numbers in Figure 1. Instead of using integers that are relatively large in both numerators and denominators, smaller integers could be used to draw a parallel. For instance, and could be used as a particular case that parallels the numbers in Figure 1. This set of numbers would be more manageable as it is not difficult to convert them to decimals without the use of a calculator. One would notice that the numbers are indeed increasing from left to right. To induce from this observed pattern, another case can be examined to confirm that this observation is indeed applicable. For example, the case of and   can be used to validate the pattern observed earlier. One would notice again that the numbers increase from left to right, thus adding credibility to the pattern observed earlier. With these two particular cases, one can therefore extend the reasoning beyond and reason inductively to reach the conclusion that the same pattern applies to the sequence of five numbers in Q1 of Figure 1.

Q2 is not much different from Q1 in terms of the content knowledge and reasoning required. As with Q1, particular cases can be considered to find a pattern. In this case, it can be observed that , , , , , etc. One would notice that the last digit of each number repeats in a pattern, 9 and 1, for every two numbers. This observed pattern can be extended beyond those given above and one can arrive at the conclusion that the last digit of is 1. However, looking for a pattern for the second last digit is more tedious as more numbers need to be written out to find a pattern as seen in Table 1 below.

Table 1. Multiples of 9

9^1 = 09	9^{11} =....09
9^2 = 81	9^{12} =....81
9^3 =..29	9^{13} =...29
9^4 =..61	9^{14} =...61
9^5 =...49	9^{15} =....49
9^6 =...41	9^{16} =....41
9^7 =...69	9^{17} =....69
9^8 =...21	9^{18} =....21
9^9 =....89	9^{19} =.....89
9^{10} =....01	9^{20} =.....01

At this juncture, it would become apparent that the last two digits repeat for every 10 numbers and the pattern repeats again after . This pattern can then be extended beyond and inductive reasoning can be performed. Therefore, for , the last two digit would occur at the fourth position, meaning that the last two digits of are the same as the last two digits of .

Both questions in Figure 1 do not require advanced mathematical content knowledge and can be classified as a Category 1 question according to Toh (2012). However, for the purposes of this paper, the questions in Figure 1 shall be categorised as “inductive reasoning type 1”, which does require any advanced mathematical content knowledge.

Another example of questions that highlight the power of inductive reasoning is shown in Figure 2, which shall be categorised as “inductive reasoning type 2”. Such questions involve additional content knowledge of basic algebraic identities or other simple results such as the laws of logarithms.

Q3.

Find the value of

$$\sqrt{1+2017\sqrt{1+2016\sqrt{1+2015\sqrt{1+2014\sqrt{1+2013\times 2011}}}}}.$$

(SMO 2017)

Q4.

Suppose that a, b and c are real numbers greater than 1. Find the value of

$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}.$$

(SMO 2009)

Figure 2. Two SMO Questions of Inductive Reasoning Type 2.

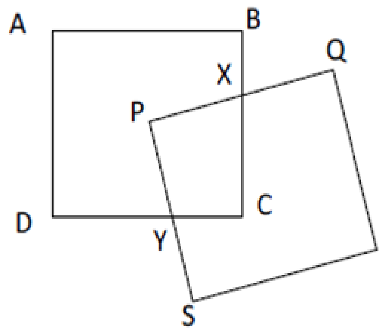
In the case of Q3, a relation needs to be made to an algebraic identity, $(a + b)(a - b) = a^2 - b^2$ before inductive reasoning can occur. 2013 can be expressed as 2012 + 1 while 2012 - 1 yields 2011. This would imply that 2013 2011 = 2012^2 - 1, which means that . The same idea can be used for which will give 2013. At this point, a clear pattern has emerged and it can be induced that the final step is .

Q4 is a very interesting question involving the knowledge of the laws of logarithm. At first glance, it seems that applying the laws of logarithm might suffice. However, on closer inspection, one would realize that elements of inductive reasoning are present also. One would be able to observe that the denominator, , in the first algebraic fraction can be simplified to using the law of logarithms. This implies that the first algebraic fraction is now . Similarly, the second algebraic fraction can be simplified to . At this point, a pattern can be observed in the way both algebraic fractions are simplified and one would notice that the denominators are while the numerators follow the pattern in the base of the logarithmic function, , in the denominator of each fraction. This pattern can be extended to the third algebraic fraction. It is evident therefore that some form of inductive reasoning is present in such a question.

A third level of inductive reasoning that can be gathered from the SMO past year questions is the idea of relating to a "special" case or considering cases. This involves finding a similarity between the given question, a particular case, and drawing a connection to another "special" case or cases where the problem can be solved in a more simplified and manageable way. Such inductive reasoning will be known as "inductive reasoning type 3" as seen in Q5 and Q6 in Figure 3.

Q5.

In the following figure, ABCD and PQRS are two identical squares with side 20cm. The point P is placed at the centre of the square ABCD. What is the area of the quadrilateral PXCY in cm^2?



(SMO 2004)

Q6.

You have 30 rods of length 5, 30 rods of length 17 and 30 rods of length 19. Using each rod at most once, how many non-congruent triangles can you form?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

(SMO 2017)

Figure 3. Two SMO Questions of Inductive Reasoning Type 3.

To simplify the problem in Q5, a "special" case where PQ is parallel to AB can be considered. In doing so, one must be able to visualize that quadrilateral PXCY has the same area as the quadrilateral formed when PX is parallel AB. This is accomplished by rotating the square PQRS clockwise about the point P until PX becomes parallel to AB. A smaller square would be obtained whose area can be calculated easily by taking cm^2. The key to using a "special" case is to identify a similarity and draw a connection from the given case to the "special" case, which is part of inductive reasoning.

In Q6, different cases need to be considered in order to find the number of non-congruent triangles that can be formed. The similarity that needs to be present is the non-congruence of the triangles formed and the fact that each rod cannot be connected to form a longer rod, as implied by the statement "using each rod at most once". Through a mere listing of the possibilities, one would discover that there are eight different triangles that can be formed. Considering the case that 3 rods are of the same length, three equilateral triangles can be formed of lengths 5 by 5 by 5, 17 by 17 by 17 and 19 by 19 by 19. The case that 2 rods are of the same length and 1 rod of a different length needs to be considered next. For that, four isosceles triangles can be formed with lengths 17 by 17 by 5, 17 by 17 by 19, 19 by 19 by 5 and 19 by 19 by 17. One more case needs to be considered where the lengths of the triangle are 5 by 17 by

19. Hence, a total of eight different triangles can be formed.

Table 2 summarizes the various types of inductive reasoning as established with reference to past year SMO questions.

Table 2. Different Types of Inductive Reasoning in SMO Questions.

Type of Inductive Reasoning	Description
Inductive Reasoning Type 1	Problems that do not require advanced mathematical content knowledge and rely largely on the process of inductive reasoning.
Inductive Reasoning Type 2	Problems that require requisite mathematical content knowledge and rely on the process of inductive reasoning.
Inductive Reasoning Type 2	Problems that require reference to a “special“ case or considering cases and rely on the process of inductive reasoning.

Implications for Teaching & Learning

Inductive reasoning is pervasively present in SMO questions as demonstrated above. It is therefore imperative that elements of inductive reasoning be made explicit in the mathematics teaching pedagogy, particularly in training students for mathematics competitions such as the SMO. On top of the requisite mathematical content knowledge, students should be taught to make observations or hypotheses about particular cases and test them with other cases. Once the hypothesis is shown to be applicable to other cases, it can be extended to all cases and a generalisation can be subsequently induced.

However, caution needs to be exercised when engaging in inductive reasoning as there could be the possibility of cases that do not conform to the generalisation induced from particular cases. This is best illustrated by examining the statement in the introductory section which states that “all birds can fly”. A counterexample needs to be found to disprove this statement made as a result of inductive reasoning. In this case, one can point out that penguins, despite being birds, do not have the ability to fly. Thus, it is pertinent to also give opportunities for students to look for counterexamples to disprove their generalisations. All that is required is a single counterexample and the conclusion derived from inductive reasoning immediately fails.

In some instances, a distinct pattern could be hard to observe due to the sheer magnitude of the numbers. An example would be Q2 in Figure 1. A sufficiently large number of cases need to be foregrounded in order to discover the pattern with the second last digit of each case. Hence, inductive reasoning might appear to be tedious but when accomplished, brings about an elegant solution to a perceived complex problem.

Conclusion

The objective of this paper is to demonstrate the importance of inductive reasoning in the SMO through showing that inductive reasoning can be used as a cognitive strategy to approach a variety of SMO questions. As highlighted above, relying solely on inductive reasoning to solve SMO questions is insufficient as a certain requisite level of mathematical content knowledge is also required. This was clearly portrayed in types 2 and 3 of inductive reasoning in Table 1. Despite that, it is undeniable that inductive reasoning is a fundamental and integral aspect of the SMO competition and mathematics in general. Therefore, there is great value in emphasizing inductive reasoning as a way of approaching mathematics problems and also as a cognitive strategy to solve mathematics problems, especially in mathematics competitions such as the SMO.

References

Carroll, J. B. (1993). *Human cognitive abilities*. *A survey of factor-analytic studies*. Cambridge, MA: Cambridge University Press.

Chua, S. K., Hang, K. H., Tay, T. S., & Teo, T. K. (2007). Singapore Mathematical Olympiads: 1995 – 2004. Singapore: Singapore Mathematical Society.

De Koning, E., Sijtsma, K., & Hamers, J. (2003). Construction and validation of a test for inductive reasoning. *European Journal of Psychological Assessment*, *19*(1), 24 - 39.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal of Research in Mathematics Education*, *22*, 170–218.

Haverty, L. A., Koedinger, K. R., Klahr, D., & Alibali, M. W. (2000). Solving inductive reasoning problems in mathematics: not-so-trivial pursuit. *Cognitive Science*, *24*(2), 249 - 298.

Howden. H. (1989). Teaching number sense. *Arithmetic Teacher*, *36*(6), 6 - 11.

Klauer, K. J. (1988). Paradigmatic teaching of inductive thinking. In H. Mandl, E. De Corte, N. Bennett, & H. F. Friedrich (Eds.), *Learning and Instruction*, *Vol. 2* (pp. 23 - 45). Oxford, UK: Pergamon Press.

Klauer, K. J. (1992). Teaching inductive reasoning to highly able children. *European Journal for High Ability*, *3*, 164 - 180.

Klauer, K. J. (1999). Fostering higher order reasoning skills: the case of inductive reasoning. In J. H. M. Hamers, J. E. H. Van Luit, & B. Csapo (Eds.), *Teaching and learning thinking skills* (pp. 131 - 155). Lisse, The Netherlands: Swets & Zeitlinger.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65 - 86). Dordrecht, The Netherlands: Kluwer Academic Publishers.

National Council of Teachers of Mathematics (NCTM). (1991). *Developing number sense in the middle grades*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nisbett, R. E., Krantz, D. H., Jepson, C., & Kunda, Z. (1983). The Use of Statistical Heuristics in Everyday Inductive Reasoning. *Psychological Review*, *90*(4), 339 - 363.

Pellegrino, J. W., & Glaser, R. (1982). Analyzing aptitudes for learning: Inductive reasoning. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 2). Hillsdale, NJ: Lawrence Erlbaum Associates.

Polya, G. (1945). *How to solve it* (2nd ed.). New York: Doubleday.

Polya, G. (1954). *Induction and analogy in mathematics: Volume I of Mathematics and plausible reasoning*. Princeton, NJ: Princeton University Press.

Sternberg, R. J., & Gardner, M. (1983). Unities in inductive reasoning. *Journal of Experimental Psychology: General*, *112*, 80–116.

Toh, T. L. (2012). *The roles of mathematics competitions in Singapore mathematics education*. Paper presented at 12th International Congress on Mathematical Education (ICME-12), July 2012, Seoul, Korea.