INFINITY AS A MATHEMATICS EDUCATION PLAYGROUND

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ABSTRACT: Infinity is a profound concept, and hence a potent educational context for triggering mathematical thinking. Building on previous research concerning the ability of students to make sense of infinity, we ask how we could think about infinity as a mathematical playground. The infinity 'equipment' we have been designing for our mathematical 'playground' now constitutes a range of inquiry-oriented activities, all at varying stages along the design-experiment continuum. For this paper, we will consider activities built from Zeno's paradoxes and Cantor's cardinality of sets. Our work with children and young adults leads us to suggest a radical position: that infinity is indeed a playground for legitimate thinking, regardless of the content outcomes of that thinking.

Keywords: mathematics instruction, mathematical experience, mathematical thinking, infinity

INTRODUCTION

Infinity – both the infinitely small and large – is a profound topic; one that can inspire paradoxes and ingenuity. Philosophers, theologians, scientists and mathematicians throughout history have grappled with the topic (Clegg, 2003; Eves, 1960). Aristotle conceived of the infinitely large as a potential, where an absolute infinity does not exist. To refute the notion of the universe as infinite, Archimedes invented base-ten exponential notation so that he could extend counting from a small measure of poppy-seeds to encompass all matter. Galileo's attempt to tame the infinitely small by describing it as immeasurably small served as a precursor to the infinitesimals developed by Newton and Leibniz and effectively critiqued by Bishop Berkeley. Modern mathematicians have developed transfinite arithmetic, in an attempt to resolve paradoxes such as Hilbert's Hotel Infinity. Infinity is a rich playground for thinking from various perspectives. This paper takes up the potential of infinity as a playground for thinking by children and young adults.

Others have considered the thinking of young adults when confronted with the profoundness of infinity, such as paradoxes that arise when trying to make sense of the infinite. This research is often framed in terms of how to shift student's intuitive notions of infinity (e.g., as a process or potential) toward accepted mathematical results (e.g., infinity as an absolute). Tsamir (2001), for example, used varying representations of the same comparing-infinite-sets task as an educational context for secondary school students to recognize inconsistencies in their thinking and thereby take-up one-to-one correspondence as a method of comparing infinite sets. (See also Meconi, 1992). This work was extended by Mamolo and Zaskis (2008), who found that intuitive notions of infinity are a conceptual barrier that university students must overcome along a learning trajectory toward resolving infinity paradoxes, such as Hilbert's Hotel Infinity, in ways consistent with current mathematical thinking.

The work of Ely (2010) moves toward the notion that infinity can be a playground for thinking, rather than a difficult concept to understand because of conceptual barriers to moving from intuitive to mathematically accepted conceptions. Ely (2010) moves in this direction when he privileges the nonstandard thinking of a university student who begins to develop notions of the infinitely small similar to those of Leibniz. This noticing of robust and non-standard thinking would not have been possible if the researcher had not temporarily set aside standard mathematical results as a target for thinking. The infinitesimal became a playground for thinking, rather than a notion to be eventually critiqued because it is not consistent with standard real analysis.

The literature noted above leads us to ask the following questions: How should we think about infinity as an educational space? What opportunities does the context of infinity provide for mathematical thinking by students of mathematics? Our work with children and young adults will lead us to suggest a more radical position than Ely: that infinity is indeed a playground for legitimate thinking, regardless of the trajectory of that thinking.

How We Got Here

This research is based is an adaptation of experimental design research, (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Shavelson, Phillips, Towne, & Feurer, 2003) a focused form of action research. Its structure involves the design and honing of instructional resources and activities that others may use. However, its purpose as a form of educational research is to explore the qualities of student understandings as the development of instructional resources progresses (Lobato, 2003). In this case, the instruction deliberately and aggressively attempts to reorient students' approaches to learning mathematics toward the challenge of developing conceptual understandings rather than the stockpiling of additional procedures, and it uses challenging mathematical activities drawn from an academic study of the history of mathematics to accomplish its educational goals.

In the past, each of us have done independent teaching experiments in which students grappled with notions of the infinite (Author A, 2010; Author B, 2009, 2005, 1992). To extend the mathematics education community's understanding of how students construct infinity as a concept, we are subjecting our stockpile of educational activities to another cycle of redesign, implementation and evaluation. In doing so, we have looked again at the student data that our previous teaching experiments generated. Fundamentally, then, our research, still in process, has two intentions, represented by two questions:

1. What features of instructional activity enable students to grapple constructively with the difficult notion of infinity?
2. What characterizes 'better' understandings of infinity as a learned mathematical concept, as students progress toward a flexible, robust, and coherent conceptual understanding?

The first of these questions will be answered, in part, by the results of our design and redesign of instructional activities, complete with materials, presented (as they must always be considered) as works-in-progress.

Answering the second question has led us to interact with the criteria that mathematics communities use to determine the appropriateness of mathematical conceptions and pose some independent considerations appropriate for a post-Tylerian (Wallin, 2002; Wright, 2000), perhaps post-constructivist (Kieren, 2000; Steffe & Kieren, 1994) orientation to the development of conceptual understanding as a curricular intention.

The Instructional Design Elements

The infinity 'equipment' we have been designing for our mathematical 'playground' now constitutes a range of inquiry-oriented activities, inclusive of activity outlines, materials, and guidelines for conversations, all at varying stages along the design-experiment continuum of design, implement, evaluate, and redesign. Fundamentally, they each take literally the challenge in William Blake's ideal:

> To see a world in a grain of sand
> And a heaven in a wild flower
> Hold infinity in the palm of your hand
> And eternity in an hour. (c. 1803)

We ask ourselves, how can students hold infinity in the palm of their hands, shape it and test it? How can they 'see' the universe, eternity, or heaven, mathematically, and question what they see? Our design elements to date (none of which live up to the scale of Blake's invitation!), include:

a. exploring three times one third in a variety of active, tactile, and visual contexts, to ask whether "point nine repeating" compares with 1.
b. repeated halving and doubling activities as tactile engagements with $(2)^n$ and $(½)^n$: fractal cards, the ancestors paradox, folding a newspaper, the radical rose, the carrot patch.
c. grappling with $y = 64/n$, as n gets really large, really small, and really nearly zero
d. reconstructing the cognition of Archimedes: the poppy-seed universe; the perimeter and area of n-gons as n is doubled; the rationality of π;
e. 'Diminishing returns' activities: The bouncing ball; iterated folding activities; taxicab-geometry sequences approaching the hypotenuse.

To illustrate the kinds of instructional activity we are designing, this paper features two other pieces of 'equipment' from our infinity playground with which readers are likely quite familiar: Zeno's paradoxes and Cantor's cardinality of sets.

Zeno's paradoxes are an educational context that could trigger thinking about the profoundness of infinity (Chavez & Reys, 2002). We used the following version: In moving from A to B, a person must first arrive at the half way point. Subsequently, the person must pass the half way point of the remaining distance. Continuing this reasoning indefinitely, a person must pass an infinite number of halfway points before reaching B. But then how is it that a person can reach B, having to traverse an infinite number of distances in a finite time? The psychological difficulty of this paradox is that we can intuitively imagine traversing an infinite number of points whereas motion is continuous and not infinitely divisible (Fishbein, 2001). In our experiment, we asked grade 4 students to act out the paradox on a learning carpet, moving from one edge to the opposite edge, following the rule that we move in steps of half the remaining distance. Students were then asked to debate whether we could reach the opposite edge of the carpet.

In the second instructional activity to be shared, students are led to engage with the comparison in size of infinite sets. Cantor's notion of the cardinality of sets has at its core the one-to-one correspondence between elements of the sets being compared (Muir, 1961). At the heart of the challenge for students is recognizing the legitimacy of one-to-one mapping as a legitimate alternative to counting as a cardinal process, even when mapping doesn't enumerate a set. The instructional design challenge begins with the notion of putting infinite sets into the hands, eyes, and minds of learners.

The instructional design for the Cantor sets inquiry features number lines made with adding machine tape. One tape shows the first twenty natural numbers {1, 2, 3, …, 20}, written every ten cm, with the rest of the natural numbers represented by "….". Another tape shows the first twenty whole numbers {0, 1, 2, 3, … 19}, and, when juxtaposed with the first tape so that equal numbers are aligned, students agree that the whole numbers are larger than the naturals by 1. When the tape is shifted, however, so that they both start together (the zero in the whole numbers is aligned with the 1 in the naturals), they are not so sure. Further sets are posed for comparison with the naturals: the naturals larger than 10; the even naturals; the multiples of 7; the squares; the negative integers. As students come to generate incompatible ideas, such as that the set of even numbers has half the numbers as the set of natural numbers, but are the same size because they can be matched/mapped one to the other, the students resolution of the disequilibrium they experience depends of their mathematical thinking and communication skills, and the leadership of their teacher.

And the Students Played

Zeno's paradox did indeed trigger the expected psychological difficulty. On the learning carpet, we saw some students finishing the remaining small distance while other students tried to creep a little bit closer. Children also would reach out with an arm and state, "I am there." We continued to remind them of the rule for taking steps (half the remaining distance), and practice the process several times, so students had experiences trying to reach the other end of the carpet several times before the class debate. During the debate, two camps emerged, one arguing that the other end of the carpet could not be reached, while the other arguing that it could be reached. Based on an articulate student representative from each camp, we found that legitimate philosophical positions emerged, one that the journey could be completed based on the indivisibility of the smallest "things" in the universe; and the other that the journey cannot be completed based on an inductive informal mathematical argument.

We are not suggesting that if this context is recreated in every classroom, then these two legitimate philosophical positions will emerge. In fact, we were surprised by a grade 4 student using an indivisible atom to defend an intuitive notion that the journey can be completed. Friedlander (2009/2010) used a Zeno-like story of a child continually eating half of a remaining cake with gifted grade 5 students. Some of these students also used an "atom is indivisible" argument to conclude that there will eventually be a last bite that consumes the rest of the cake. Friedlander used Zeno-like stories as an educational context for children to informally explore their intuitive conceptions of infinity, where these intuitions are a starting point for moving toward taking-up accepted formal mathematical understandings of infinity in later grades. Our data suggests that these intuitions are in fact legitimated by the philosophical qualities of their argumentation. These qualities could not have been noticed, let alone triggered if the disposition of the teacher was not one of legitimating thinking, regardless of whether it would eventually converge on accepted truths.

Similarly, we were able to lead students to engage with the cardinality of infinite sets effectively. Students perceived the entry points to the inquiry as accessible and inviting. They were able to arrive at cognitive positions in which two apparently irreconcilable answers both appeared credible. They were able to generate and explore potential resolutions of their disequilibrium. To some extent they were able to see beyond the limitations of their newly generated understandings toward the frontiers of mathematics that Cantor explored. Some students felt it was a matter of perception: "it depends on how you look at it." Others felt that there had to be something wrong, because "they can't both be right. There's only one right answer, in math." After discussing the examples they shared, the students expected to be told the resolution to this riddle. We did not satisfy their expectation, and so they left the infinity playground having been happily perplexed by their experiences but disappointed to have no resolution.

What We are Learning

We are discovering that it is not impossible to enable students to construct the infinite and infinitesimals with their hands, their eyes, and their minds. Like children on playground equipment, students have explored the possibilities of the equipment we provide and the activities we recommend, and have been intrigued by the dissonance that their inquiries generate. We have found that the design of the playground equipment matters, and we are fine-tuning our design through each iteration we plan. For instance, in our last enactment of infinity, students determined the halfway point between their position and their goal by visual estimation. We are considering having them use string to measure the distance without numbers, and half that distance by folding the string. How will the increased accuracy affect their conceptions of what happens as the distances get very small? We look forward to finding out. At the same time, we are wondering what might be gained by having students look at diminishing sequences in more than one context concurrently—would students recognize similarities and differences between their Zeno experiences and an experience determining the distance traveled by a dropped rubber ball (Vinogradova & Blaine, 2010)?

Similarly, with the Cantor sets, we continue to fine-tune. We would like to find out the effects on students' generalization across cases if groups each get to pioneer a different set comparison by building the paper-tape number lines themselves for follow-up comparisons. If our group explores comparisons of the multiples of five to the natural numbers, and another group compares the powers of two to the even numbers, will students appreciate the experiences and interpretations from other groups in comparison to their own? In our design research, activity design is a form of playground for us as educators, and we're having fun.

Yet there are bigger ideas emerging from our inquiry. We began by considering what students could learn about infinity. We have drifted, not accidentally, toward discussing what students might experience by engaging with infinity. Through interacting with the thinking by students, we have come to perceive it to be legitimate and valuable in and of itself, regardless of the consistency of the students' conclusions with current mathematical resolutions regarding the paradoxes of infinity. The thinking of students described above is a potent reminder to us that thinking is a goal that should trump all other considerations, including the current canons of mathematics. Such a conclusion leads to curricular and research recommendations.

In terms of curriculum, that educators would want mathematical learners to eventually take-up – understand and accept – current truths in mathematics cannot eclipse our desire to have students engage in thinking as a process. We find ourselves shifting our attention away from content or outcomes respectful of the continually evolving mathematical canon concerning infinity. As Ely (2010) suggests, infinity as a playground can provide students with opportunities to exercise their thinking processes, increasing their capacity to learn content.

Further, we believe that a rich line of educational research is available by generating activities where students play with infinity. As such, our future work will continue to develop contexts for exploring the infinity playground. For example, what qualities of thinking will arise when students try to make sense of zero divided by zero ("0/0=?"), where the educational context does not privilege in any way the current mathematical thinking that 0/0 is indeterminate? Considering $0/n$ for $n \neq 0$ suggests $0/0 = 0$. Considering n/n for $n \neq 0$ suggests that $0/0 = 1$. Considering $1/n$ as n approaches 0 suggests 0/0 = infinity. Considering 1/0 when entered into a calculator suggests that 0/0 is out of bounds, sinful in some way (Beskwick, Richman, 1999). We suggest that *educationally* it is less relevant how much mathematics content students remember about this mathematics, or whether their conclusions map onto Dubinsky's notions of 'actual' infinity (Mamalo & Zazkis, 2008). *Educationally,* what matters more is how, and how much, and how well, students develop as mathematical thinkers as they engage with each piece of equipment on the mathematical playground. And we believe that, in terms of mathematics education, the pedagogy of infinity may be remarkable 'playground equipment' for research about mathematical cognition, the growth of mathematical understandings through experience.

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