An Introduction to Pade Approximation

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The Pade approach has emerged based on continued fractions and the Euclidean algorithm, and its history dates back to ancient times. The Pade approach continues to attract the attention of researchers with the development of high-speed computers and the application of the Pade approach to numerical analysis, fluid mechanics, theoretical physics, signal processing, cryptography, engineering and other disciplines (Andrianov & Shatrov, 2020; Brezinski, 1990; Wuytack, 2006). That's why it's still popular.

Pade approximations are widely used in numerical solutions of differential equations, analytical continuation problems of power series, orthogonal polynomials, rational approximations, Gaussian quadrature, convergence acceleration, computation of special functions, finding singularity, zero points, roots and poles of power series (Andrianov & Shatrov, 2020; Brezinski, 1990; Wuytack, 2006).

Most solutions of ordinary or partial differential equations are of power series forms due the result of numerical solutions of the associated differential equations upon approximation or truncation. These power series are often approximated by polynomials, nevertheless, polynomials tend to exhibit oscillations that may produce error bounds, also, the singularities of polynomials cannot be observed clearly in a finite plane (Peker et al., 2011; Wazwaz, 2002; Wazwaz, 2009). Therefore, Pade approximants yield meromorphic continuations of functions defined by power series and can be used even in cases where analytic continuations are inapplicable. If a Pade approximation converges to the given function, then roots of the denominator tend to points of singularities. One-point Pade approximants give possibilities to improve convergence of series (Gonchar, 1986; Litvinov, 1994; Litvinov, 2003; Suetin, 2002; Suetin, 2004).

A Pade approximant of a function is a rational function of two polynomial functions where the coefficients of the numerator and the denominator depend on the coefficients of the concerned function (Baker & Graves-Moris, 1996; Boyd, 1979; Wazwaz, 2009).

Let f be a function analytic on a neighborhood of $x=0$ and let n and m be non-negative integers. In the literature (Baker & Graves-Moris, 1996; Brezinski, 1990), Pade approximant of f with order (n,m) where $n,m \in \mathbb{N} \cup \{0\}$, denoted by $f_{[n/m]}(x)$, is defined as the unique rational function

$$f_{[n/m]}(x) = \frac{P_n(x)}{Q_m(x)}$$

where $\deg P_n(x) \le n$, $\deg Q_m(x) \le m$ and $Q_m(x) \not\equiv 0$.

Suppose that the power series of f is given as

$$f(x)=\sum_{k=0}^{\infty}c_k x^k$$

where c_k's are non-negative. For simplicity, we assume that the power series is about zero. Now, the principle is that the formal series,

$$Q_m(x)f_{[n/m]}(x)-P_n(x)$$

coincides with f as far as possible. That is, our aim is to make the maximum error as small as possible (Mathews & Fink, 2004). In other words, it is required that

$$Q_m(x)f_{[n/m]}(x)-P_n(x)=O\left[x^{n+m+1}\right]$$

where $O\left[x^t\right]$ denotes for some power series of the form $\sum_{k=t}^{\infty}c_k x^k$.

Pade approximants can be written as the following expression (Baker & Graves-Morris, 1996)

$$f_{[n/m]}(x)=\frac{p_0+p_1x+p_2x^2+\ldots+p_nx^n}{q_0+q_1x+q_2x^2+\ldots+q_mx^m}$$

where q_0 can be taken 1 so that $Q_m(x)\not\equiv 0$. Additionally, numerator and denominator have no common factors (Wazwaz, 2002).

By using the last equation, the coefficients of polynomials $P_n(x)$ and $Q_m(x)$ can be determined as follows:

$$P_n(x)=p_0+p_1x+p_2x^2+\ldots+p_nx^n,$$

$$Q_m(x)=q_0+q_1x+q_2x^2+\ldots+q_mx^m.$$

There are $n+1$ numerator coefficients and $m+1$ denominator coefficients. Thus, $n+m+1$ unknown coefficients in all. Since $f(x)=\sum_{k=0}^{\infty}c_k x^k$, the series can be written as

$$f(x)=\sum_{k=0}^{\infty}c_k x^k=\frac{p_0+p_1x+p_2x^2+\ldots+p_nx^n}{1+q_1x+q_2x^2+\ldots+q_mx^m}+O\left(x^{n+m+1}\right).$$

By cross-multiplying last equation, one can find that

$$\left(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\ldots\right)\left(1+q_1x+q_2x^2+\ldots+q_mx^m\right)$$

$$=\left(p_0+p_1x+p_2x^2+\ldots+p_nx^n\right)+O\left(x^{n+m+1}\right)$$

The following set of equations are deduced from this equation:

$$
\begin{aligned}
c_0 &= p_0 \\
c_1 + c_0 q_1 &= p_1 \\
c_2 + c_1 q_1 + c_0 q_2 &= p_2 \\
\vdots \quad & \vdots \quad \vdots \\
c_n + c_{n-1} q_1 + \ldots + c_0 q_m &= p_n
\end{aligned}
$$

and

$$
\begin{aligned}
c_{n+1} + c_n\, q_1 + \ldots + c_{n-m+1}\, q_m &= 0 \\
c_{n+2} + c_{n+1}\, q_1 + \ldots + c_{n-m+2}\, q_m &= 0 \\
\vdots \quad & \vdots \quad \vdots \\
c_{n+m} + c_{n+m-1}\, q_1 + \ldots + c_n\, q_m &= 0
\end{aligned}
$$

where $c_n = 0$ for $n < 0$ and $q_j = 0$ for $j > m$.

The solution, that is $f_{[n/m]}(x)$, can be written in the following form

$$
f_{[n/m]}(x) = \frac{\begin{vmatrix} c_{n-m+1} & c_{n-m+2} & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{n+m} \\ \sum\limits_{j=m}^{n} c_{j-m}\, x^j & \sum\limits_{j=m-1}^{n} c_{j-m+1}\, x^j & \cdots & \sum\limits_{j=0}^{n} c_j\, x^j \end{vmatrix}}{\begin{vmatrix} c_{n-m+1} & c_{n-m+2} & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{n+m} \\ x^m & x^{m-1} & \cdots & 1 \end{vmatrix}},
$$

(Baker & Graves - Moris, 1996; Boyd, 1979; Wazwaz, 2009**).**

Example 1: Find the Pade approximation $f_{[2/2]}(x)$ for $f(x) = e^x$.
Solution:
Maclaurin series expansion of the given function is as follows:

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O\left(x^5\right).$$

Since $m = n = 2$,

$$P_2(x) = p_0 + p_1 x + p_2 x^2 \quad \text{and} \quad Q_2(x) = q_0 + q_1 x + q_2 x^2 = 1 + q_1 x + q_2 x^2.$$

$$f(x) = \frac{P_2(x)}{Q_2(x)} \Rightarrow f(x)\, Q_2(x) - P_2(x) = 0$$

$$\Rightarrow \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O\left(x^5\right)\right)\left(1 + q_1 x + q_2 x^2\right) - \left(p_0 + p_1 x + p_2 x^2\right) = 0$$

$$\Rightarrow (1-p_0)+(1-p_1+q_1)x+\left(\frac{1}{2}-p_2+q_1+q_2\right)x^2+\left(\frac{1}{6}+\frac{q_1}{2}+q_2\right)x^3$$

$$+\left(\frac{1}{24}+\frac{q_1}{6}+\frac{q_2}{2}\right)x^4+O\left(x^5\right)=0$$

By equating to zero both sides of the last equation, the following equations find:

$$1-p_0=0,$$

$$1-p_1+q_1=0,$$

$$\frac{1}{2}-p_2+q_1+q_2=0,$$

$$\frac{1}{6}+\frac{q_1}{2}+q_2=0,$$

$$\frac{1}{24}+\frac{q_1}{6}+\frac{q_2}{2}=0.$$

Solving these equations simultaneously, coefficients of $P_2(x)$ and $Q_2(x)$ can be found as follows:

$$p_0=1,$$

$$p_1=\frac{1}{2},$$

$$p_2=\frac{1}{12},$$

$$q_1=-\frac{1}{2},$$

$$q_2=\frac{1}{12}.$$

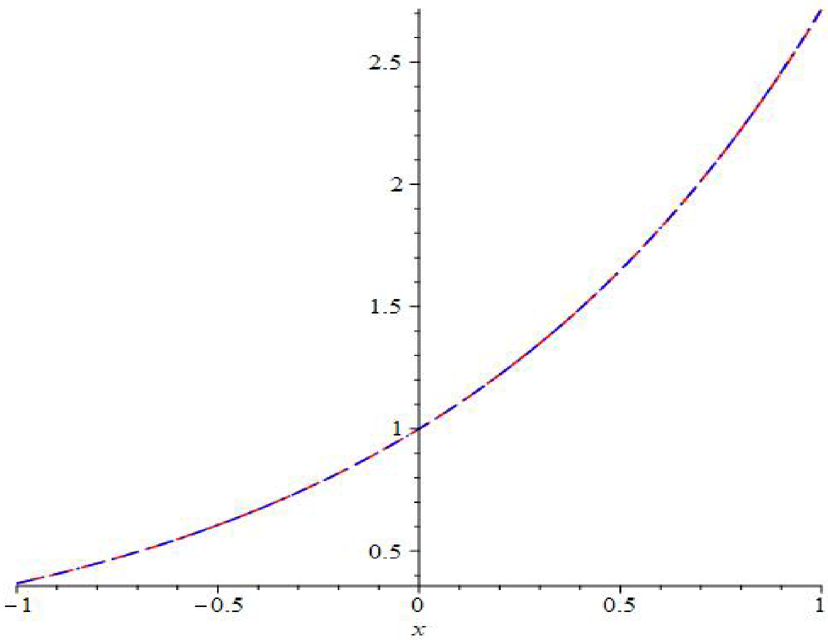
Therefore, $[2/2]$ Pade approximation of $f(x)=e^x$ is as follows:

$$f_{[2/2]}(x)=\frac{1+\frac{1}{2}x+\frac{1}{12}x^2}{1-\frac{1}{2}x+\frac{1}{12}x^2}=\frac{12+6x+x^2}{12-6x+x^2}.$$

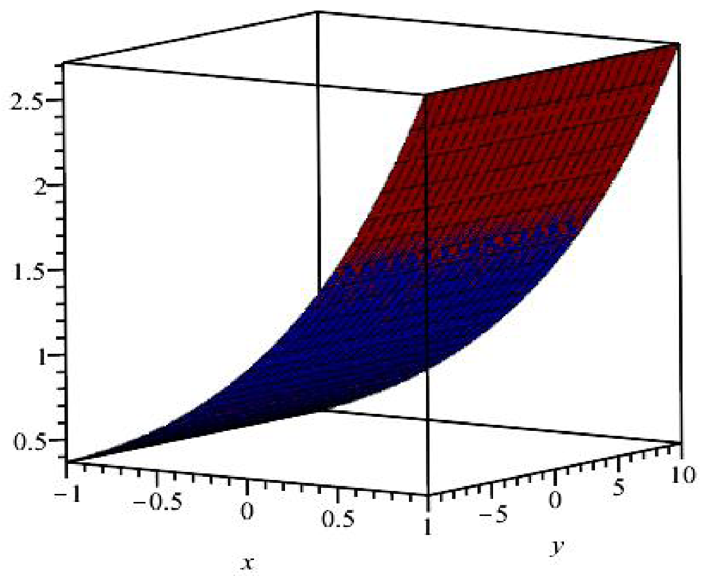
By a similar manner, one can easily find the following Pade approximation table of $f(x)=e^x$ when $m=0,1,2,3$ and $n=0,1,2,3$.

n/m	0	1	2	3
0	$\frac{1}{1}$	$\frac{1}{1-x}$	$\frac{1}{1-x+\frac{1}{2}x^2}$	$\frac{1}{1-x+\frac{1}{2}x^2-\frac{1}{6}x^3}$
1	$\frac{1+x}{1}$	$\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}$	$\frac{1+\frac{1}{3}x}{1-\frac{2}{3}x+\frac{1}{6}x^2}$	$\frac{1+\frac{1}{4}x}{1-\frac{3}{4}x+\frac{1}{4}x^2-\frac{1}{24}x^3}$
2	$\frac{1+x+\frac{1}{2}x^2}{1}$	$\frac{1+\frac{2}{3}x+\frac{1}{6}x^2}{1-\frac{1}{3}x}$	$\frac{1+\frac{1}{2}x+\frac{1}{12}x^2}{1-\frac{1}{2}x+\frac{1}{12}x^2}$	$\frac{1+\frac{2}{5}x+\frac{1}{20}x^2}{1-\frac{3}{5}x+\frac{3}{20}x^2-\frac{1}{60}x^3}$
3	$\frac{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}{1}$	$\frac{1+\frac{3}{4}x+\frac{1}{4}x^2+\frac{1}{24}x^3}{1-\frac{1}{4}x}$	$\frac{1+\frac{3}{5}x+\frac{3}{20}x^2+\frac{1}{60}x^3}{1-\frac{2}{5}x+\frac{1}{20}x^2}$	$\frac{1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{120}x^3}{1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3}$

The graph of the given function and its Pade approximation over the interval $[-1,1]$ can be drawn by using MAPLE as below:



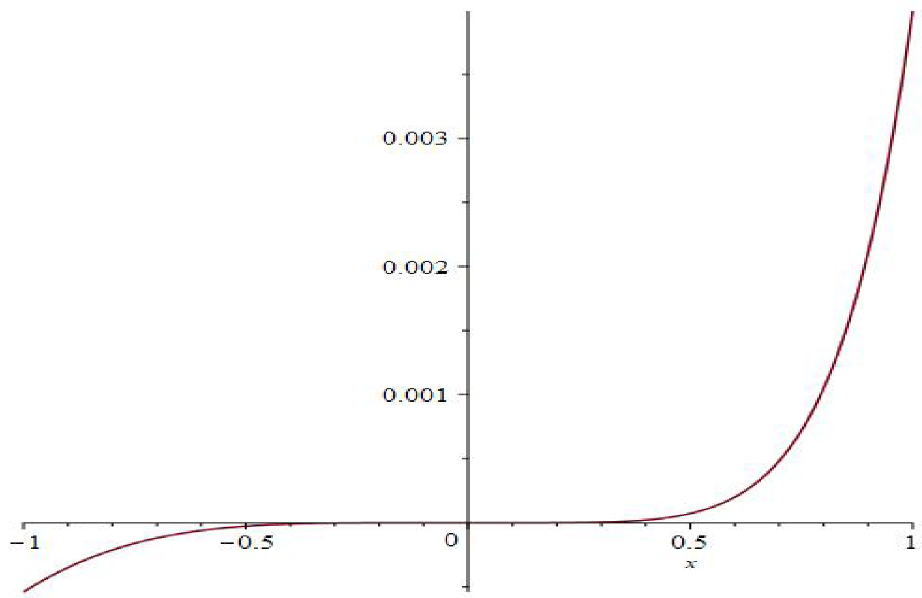
The three-dimensional graph of the given function and its Pade approximation over the interval $[-1,1]$ can be drawn by using MAPLE as follows:



The error between the given function and its Pade approximation is given by

$$f(x)-f_{[2/2]}(x)=e^{x}-\frac{12+6x+x^{2}}{12-6x+x^{2}}.$$

The graph of the error over the same interval can be plotted as follows:



The table given below shows the error over the same interval.

x	**Error** $\left(f(x)-f_{[2/2]}(x)\right)$
-1	-0.0005416114
-0.8	-0.0002123204
-0.6	-0.0000605444
-0.4	-0,0000096245
-0.2	-0.0000003649
0	0
0.2	0.000000544
0.4	0.000021420
0.6	0.000200992
0.8	0.001051132
1	0.003996114

It is easily seen from the table that the maximum error is

$$\left|f(x)-f_{[2/2]}(x)\right| \leq 0.003996114 .$$

Example 2: Find the Pade approximations $f_{[2/2]}(x)$, $f_{[3/3]}(x)$ and $f_{[4/4]}(x)$ for the function $f(x)=\tanh x$.

Solution:

Maclaurin series expansion of the given function is as follows:

$$f(x)=\tanh x = x-\frac{1}{3}x^3+\frac{2}{15}x^5-\frac{17}{315}x^7+O\left(x^8\right).$$

First of all, we will find $f_{[2/2]}(x)$. Since $m=n=2$,

$$P_2(x)=p_0+p_1x+p_2x^2 \quad \text{and} \quad Q_2(x)=q_0+q_1x+q_2x^2=1+q_1x+q_2x^2 .$$

$$f(x)=\frac{P_2(x)}{Q_2(x)} \Rightarrow f(x)\,Q_2(x)-P_2(x)=0 .$$

$$\Rightarrow\left(x-\frac{1}{3}x^3+\frac{2}{15}x^5-\frac{17}{315}x^7+O\left(x^8\right)\right)\left(1+q_1x+q_2x^2\right)-\left(p_0+p_1x+p_2x^2\right)=0$$

$$\Rightarrow(-p_0)+(1-p_1)x+(q_1-p_2)x^2+\left(q_2-\frac{1}{3}\right)x^3-\frac{q_1}{3}x^4+O\left(x^5\right)=0$$

By equating to zero both sides of the last equation, the following equations find:

$$-p_0=0$$

$$1-p_1=0$$

$$q_1-p_2=0$$

$$q_2 - \frac{1}{3} = 0$$

$$\frac{-q_1}{3} = 0$$

Solving these equations simultaneously, coefficients of $P_2(x)$ and $Q_2(x)$ can be found as follows:

$$p_0 = 0,\ p_1 = 1,\ p_2 = 0$$

$$q_1 = 0,\ q_2 = \frac{2}{5}.$$

Thus, $[2/2]$ Pade approximation of $f(x) = \tanh x$ is obtained as follows:

$$f_{[2/2]}(x) = \frac{x}{1+\frac{1}{3}x^2} = \frac{3x}{3+x^2}.$$

Secondly, in order to find $f_{[3/3]}(x)$, since $m = n = 3$, we have $P_3(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

and $Q_3(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 = 1 + q_1 x + q_2 x^2 + q_3 x^3$.

$$f(x) = \frac{P_3(x)}{Q_3(x)} \Rightarrow f(x)\,Q_3(x) - P_3(x) = 0.$$

$$\Rightarrow \left(x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + O\left(x^9\right)\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3\right) - \left(p_0 + p_1 x + p_2 x^2 + p_3 x^3\right) = 0$$

By a similar manner as above, coefficients of $P_3(x)$ and $Q_3(x)$ can be found as follows:

$$p_0 = 0,\ p_1 = 1,\ p_2 = 0,\ p_3 = \frac{1}{15}$$

$$q_1 = 0,\ q_2 = \frac{2}{5},\ q_3 = 0.$$

Thus, $[3/3]$ Pade approximation of $f(x) = \tanh x$ is obtained as follows:

$$f_{[3/3]}(x) = \frac{x + \frac{1}{15}x^3}{1+\frac{2}{5}x^2} = \frac{15x + x^3}{15 + 6x^2}.$$

Finally, $f_{[4/4]}(x)$ will be found as follows, since $m = n = 4$, we have $P_4(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4$

and $Q_3(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 = 1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4$.

$$f(x) = \frac{P_4(x)}{Q_4(x)} \Rightarrow f(x)\,Q_4(x) - P_4(x) = 0.$$

$$\Rightarrow \left(x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + O\left(x^9\right)\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4\right) - \left(p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4\right) = 0$$

By a similar manner as above two stages, coefficients of $P_4(x)$ and $Q_4(x)$ can be found as follows:

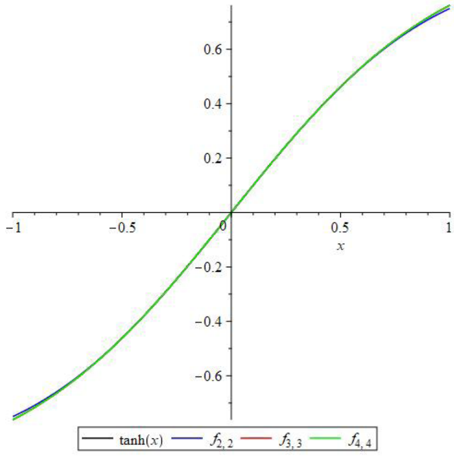
$$p_0 = 0,\ p_1 = 1,\ p_2 = 0,\ p_3 = \frac{2}{21},\ p_4 = 0$$

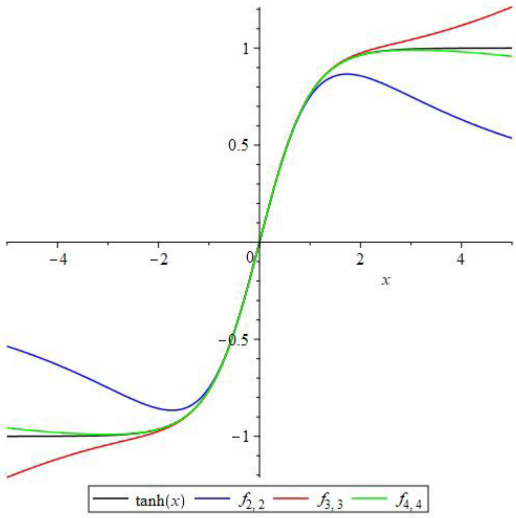
$$q_1 = 0,\ q_2 = \frac{3}{7},\ q_3 = 0,\ q_4 = \frac{1}{105}.$$

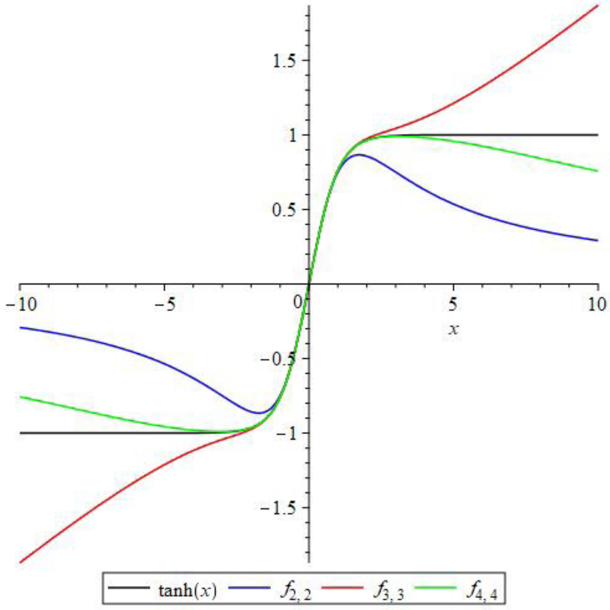
Hence, $[4/4]$ Pade approximation of $f(x) = \tanh x$ is obtained as follows:

$$f_{[4/4]}(x) = \frac{x + \frac{2}{21}x^3}{1 + \frac{3}{7}x^2 + \frac{1}{105}x^4} = \frac{105x + 10x^3}{105 + 45x^2 + x^4}.$$

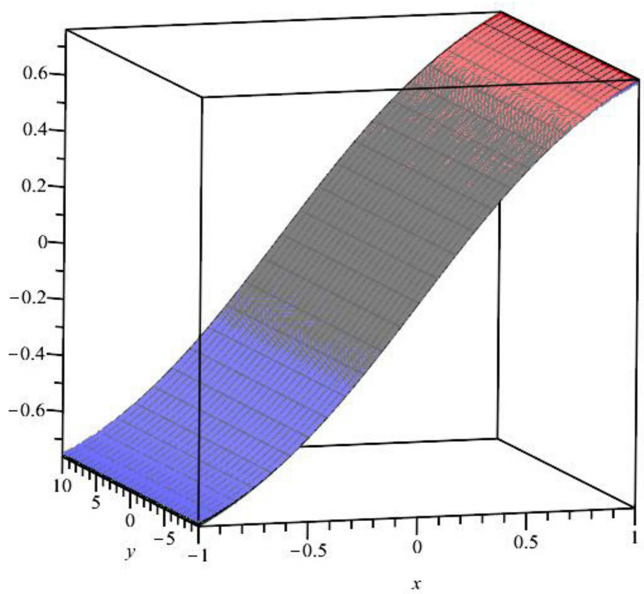
The graph of the given function and its Pade approximation over the intervals $[-1,1]$, $[-5,5]$ and $[-10,10]$ can be drawn by using MAPLE as below:

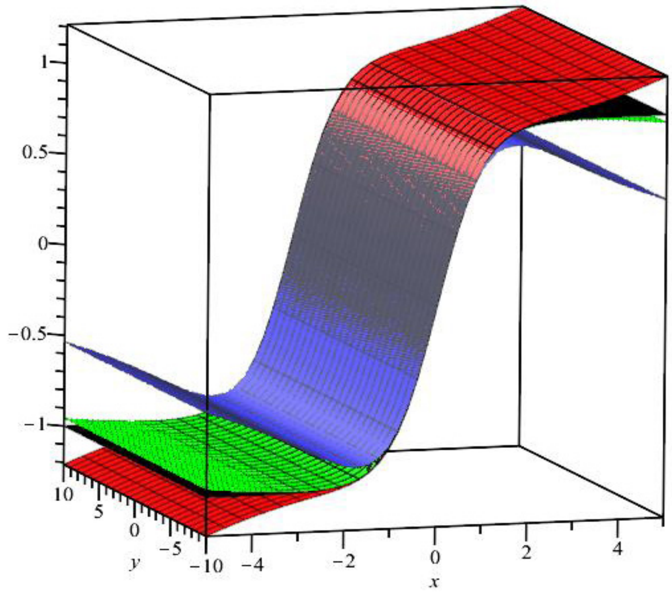


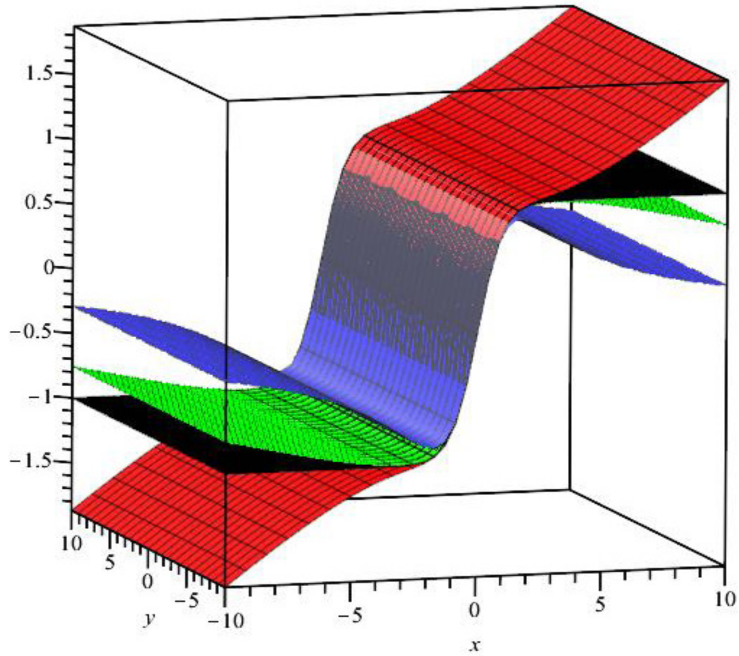




The three-dimensional graph of the given function and its Pade approximations over the intervals $[-1,1]$, $[-5,5]$ and $[-10,10]$ can be drawn by using MAPLE as follows:







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