Continued Fractions and Pell's Equation

Bilge PEKER
Necmettin Erbakan University

1.Introduction

The study of Diophantine equations is one of the most important topics in the history of number theory. Diophantine equation is an $n-$variable $(n \geq 2)$ polynomial equation $f(x_1, x_2, x_3, \ldots, x_n) = 0$ whose coefficients are integers.

The studies related to Diophantine equations are based on three fundamental problems (Andreescu et al., 2010):

- The first one is whether the Diophantine equation is solvable.
- The second one if it is solvable, is the number of solutions to the Diophantine equation finite or infinite?
- The third problem is that if the Diophantine equation is solvable, determine all of its solutions.

The form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ where $a, b, c, d, e, f \in \mathbb{R}$ and $x, y \in \mathbb{Z}$ is called as quadratic Diophantine equation. Such an equation with integral coefficients is reduced in its main case to Pell-type equation. So, a Pell's equation is a kind of Diophantine equation.

Pell's equation has a long history. Many mathematicians have been fascinated by Pell's equations and have done a lot of work on it. The first important development regarding the solution of Pell's equations was in India. In AD 628, Brahmagupta explained how to use known solutions of Pell's equation to generate new solutions. After that in AD 1150 Bhaskaracharya gave a method for finding a minimal positive solution to Pell's equation. Brahmagupta explains a method for generating new solutions from old ones and gives an algorithm. After the years Bhaskaracharya extended Brahmagupta's work on Pell's equation via repeated reductions. Bhaskaracharya showed his method by solving the equation $x^2 - 61y^2 = 1$. In AD 1657, Fermat challenged his fellow mathematicians to solve the equation $x^2 - 61y^2 = 1$, and thus began the modern European history of the Pell equation. Brouncker gave a general method for solving Pell's equation and solved the equation $x^2 - 313y^2 = 1$ (Silverman, 2013). John Wallis described Brouncker's method in his book entitled Opera Mathematica. Euler mistakenly thought that the method in Wallis's book was created by John Pell and this name was given to the Pell's equation by

Euler. Therefore, the quadratic Diophantine equation of the form $x^2 - Dy^2 = 1$, where D is a positive non-square integer, unknowns x and y are positive integers, is called as *Pell's equation*, following an erroneous attribution of Euler. Brouncker and Wallis explained a method of solution that is the same as the solution by continued fractions (Coppel, 2006).

Many great mathematicians of the seventeenth and eighteenth centuries have been fascinated by continued fractions and have done work on it. Appearing in many areas of mathematics, continued fractions are interesting and useful in other areas of number theory. For instance, continued fractions are "the best" approximations of real numbers. Continued fractions, which also provide a way to learn about the decimal approximations of rational numbers, also appear in many other areas. Additionally, continued fractions provide a way to analyze solutions to Pell's equation $x^2 - Dy^2 = 1$. Since all integral solutions of Pell's equation come from convergents to $\sqrt{D}$.

2. Continued Fractions

Definition 2.1. (Olds, 1963) An expression of the following form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ddots}}}$$

is called as *continued fraction* where the $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are any real or complex numbers, and the number of terms is finite or infinite.

The purpose of the present section is to acquaint with the so-called regular continued fractions, that is, those of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

usually with the assumption that all the elements $a_1, a_2, a_3, \ldots,$ are positive integers.

Definition 2.2. (Rosen, 1992) An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is called as a *finite continued fraction* where $a_0, a_1, a_2, \ldots, a_n$ are real numbers with $a_1, a_2, \ldots, a_n$ positive. A finite continued fraction is denoted by $[a_0; a_1, a_2, \ldots, a_n]$ where the real numbers $a_1, a_2, \ldots, a_n$ are called the partial quotients of the continued fraction. The continued fraction is called *simple* if the real numbers $a_0, a_1, a_2, \ldots, a_n$ are all integers.

A finite continued fraction can also be written as $[a_0; a_1, a_2, \ldots, a_n] = a_0 + \dfrac{1}{[a_1; a_2, \ldots, a_n]} = [a_0; [a_1, a_2, \ldots, a_n]]$ for $n > 0$.

Example 2.3. Express $[2;1,3,1,4]$ as a rational number.

$$[2;1,3,1,4] = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}} = \frac{67}{24}$$

As can be seen, the value of any finite simple continued fraction is always a rational number and every rational number can be represented by a finite simple continued fraction (Rosen, 1992; Burton, 2010; Robbins, 1993).

Example 2.4. Express $\dfrac{67}{29}$ as a finite simple continued fraction.

By the Euclidean Algorithm, we have

$$67 = 2.29 + 9$$

$$29 = 3.9 + 2$$

$$9 = 4.2 + 1$$

$$2 = 2.1 + 0,$$

it follows that

$$\frac{67}{29} = 2 + \frac{9}{29} = 2 + \frac{1}{29/9} = 2 + \cfrac{1}{3 + \cfrac{2}{9}}$$

$$= 2 + \cfrac{1}{3 + \cfrac{1}{9/2}} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

$$= [2;3,4,2].$$

Since $2 = 1 + 1$, it can be written

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

Therefore, it can also be denoted as $[2;3,4,1,1]$.

This explains the following theorem.

Theorem 2.5. (Long, 1987) If $a_n > 1$, then $[a_0;a_1,a_2,\ldots,a_n] = [a_0;a_1,a_2,\ldots,a_n - 1,1]$.

Definition 2.6. (Rosen, 1992) Let $A = [a_0;a_1,a_2,\ldots,a_n]$ where $\forall a_i \in \mathbb{R}$ with $a_1,a_2,\ldots,a_n$ positive. The continued fractions $C_k = [a_0;a_1,a_2,\ldots,a_k]$, where $k \in \mathbb{Z}$ with $0 \le k \le n$, is defined as the *kth convergent of the continued fraction* $A = [a_0;a_1,a_2,\ldots,a_n]$ and it is denoted by C_k.

Theorem 2.7. (Stein, 2008) If real numbers p_k and q_k are defined as follows:

$p_{-2} = 0$, $p_{-1} = 1$, $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, … $p_k = a_k p_{k-1} + p_{k-2}$ …,

$q_{-2} = 1$, $q_{-1} = 0$, $q_0 = 1$, $q_1 = a_1$, … $q_k = a_k q_{k-1} + q_{k-2}$, …,

then the *kth convergent* $C_k = [a_0;a_1,a_2,\ldots,a_k]$ is given by $C_k = \frac{p_k}{q_k}$ for $0 \le k \le n$.

Theorem 2.8. (Burton, 2011)

a. The convergents with even subscripts form a strictly increasing sequence; that is, $C_0 < C_2 < C_4 < \ldots$.

b. The convergents with odd subscripts form a strictly decreasing sequence; that is, $C_1 > C_3 > C_5 > \ldots$.

c. Every convergent with an odd subscript is greater than every convergent with an even subscript.

In other words, this theorem briefly states that $C_0 < C_2 < C_4 < \ldots < C_n < \ldots < C_5 < C_3 < C_1$.

Theorem 2.9. (Long, 1987) Let $\alpha = [a_0;a_1,a_2,\ldots,a_n]$ with $a_n > 1$ so that α is the rational number $\frac{p_n}{q_n}$. Then, for $1 \le i \le n$, we have that

$$\left|\alpha - \frac{p_i}{q_i}\right| < \left|\alpha - \frac{p_{i-1}}{q_{i-1}}\right|$$

and also

$$|\alpha q_i - p_i| < |\alpha q_{i-1} - p_{i-1}|.$$

Let's show what has been given so far on an example.

Example 2.10. Express $\frac{170}{39}$ as a finite simple continued fraction and compute the convergents for this simple continued fraction. Also, show that its continued fraction satisfies Theorem 2.5., Theorem 2.8. and Theorem 2.9.

By the Euclidean Algorithm, we have

$$\begin{aligned} 170 &= 4.39+14 \\ 39 &= 2.14+11 \\ 14 &= 1.11+3 \\ 11 &= 3.3+2 \\ 3 &= 1.2+1 \\ 2 &= 2.1+0, \end{aligned}$$

it follows that

$$\frac{170}{39} = 4+\frac{14}{39} = 4+\frac{1}{39/14} = 4+\cfrac{1}{2+\cfrac{11}{14}}$$

$$= 4+\cfrac{1}{2+\cfrac{1}{14/11}} = 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{3}{11}}}$$

$$= 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{11/3}}} = 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{2}{3}}}}$$

$$= 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3/2}}}} = 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2}}}}}$$

$$=[4;2,1,3,1,2].$$

Since $2=1+1$, it can be written

$$\frac{170}{39}=4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}}$$

Therefore, it can also be denoted as $[4;2,1,3,1,1,1]$. This satisfies Theorem 2.5.

The various convergents are

$C_0=[4]$ $\qquad C_0=\frac{p_0}{q_0}=\frac{4}{1}=4$

$C_1=[4;2]$ $\qquad C_1=\frac{p_1}{q_1}=\frac{9}{2}=4,5$

$C_2=[4;2,1]$ $\qquad C_2=\frac{p_2}{q_2}=\frac{13}{3}\approx 4,3333333333$

$C_3=[4;2,1,3]$ $\qquad C_3=\frac{p_3}{q_3}=\frac{48}{11}\approx 4,3636363636$

$C_4=[4;2,1,3,1]$ $\qquad C_4=\frac{p_4}{q_4}=\frac{61}{14}\approx 4,3571428571$

$C_5=[4;2,1,3,1,2]$ $\qquad C_5=\frac{p_5}{q_5}=\frac{170}{39}\approx 4,358974359$.

It is clear that $\frac{4}{1}<\frac{13}{3}<\frac{61}{14}<\frac{170}{39}<\frac{48}{11}<\frac{9}{2}$. Thus, $\frac{p_0}{q_0}<\frac{p_2}{q_2}<\frac{p_4}{q_4}<\frac{p_5}{q_5}<\frac{p_3}{q_3}<\frac{p_1}{q_1}$.
Therefore, $C_0<C_2<C_4<C_5<C_3<C_1$. This satisfies Theorem 2.8.

Let us check that if Theorem 2.9 is satisfied.

$$\left|\frac{170}{39}q_4-p_4\right|=\left|\frac{170}{39}14-61\right|=\left|\frac{1}{39}\right|\approx 0,0256410256$$

$$\left|\frac{170}{39}q_3-p_3\right|=\left|\frac{170}{39}11-48\right|=\left|\frac{-2}{39}\right|\approx 0,0512820513$$

From this we easily obtain $\left|\frac{170}{39}q_4-p_4\right|<\left|\frac{170}{39}q_3-p_3\right|$. It can be shown similarly for the others p_k and q_k $(1\le k\le 5)$.

Theorem 2.11. (Koshy, 2007) Let $C_k=\frac{p_k}{q_k}$ be the *kth convergent* of the simple continued fraction $[a_0;a_1,a_2,\ldots,a_n]$ where $1\le k\le n$. Then, $p_kq_{k-1}-q_kp_{k-1}=(-1)^{k-1}$ is valid.

The Procedure of Continued Fraction

The procedure of continued fraction can also be explained as follows.

Let $x \in \mathbb{R}$ and

$$x = \lfloor x \rfloor + \{x\} = a_0 + \{x\}$$

where $\lfloor x \rfloor \in \mathbb{Z}$ and $0 \leq \{x\} < 1$.

If $x \in \mathbb{Z}$, then this is the end of the algorithm.

If $x \notin \mathbb{Z}$, i.e. $\{x\} \neq 0$, then we write $x_1 = \frac{1}{\{x\}}$. Therefore

$$x = \lfloor x \rfloor + \frac{1}{x_1} \text{ with } x_1 > 1.$$

If $x_1 \in \mathbb{Z}$, then this is the end of the algorithm.

If $x_1 \notin \mathbb{Z}$, then we write $x_2 = \frac{1}{\{x_1\}}$. Therefore,

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{x_2}} \text{ with } x_2 > 1.$$

Set $a_0 = \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor$ for $i \geq 1$.

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{\lfloor x_2 \rfloor + \cfrac{1}{\cdots}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

Consequently, $x = [a_0; a_1, a_2, \ldots]$. The algorithm finishes after finitely many steps if and only if x is rational.

Example 2.12. Let $x = \frac{24}{7}$. Then $x = 3 + \frac{3}{7}$ i.e. $a_0 = 3$ and $\{x\} = \frac{3}{7}$.

$x_1 = \frac{1}{\{x\}} = \frac{7}{3} = 2 + \frac{1}{3}$, so $a_1 = 2$ and $\{x_2\} = \frac{1}{3}$.

$x_3 = \frac{1}{\{x_2\}} = \frac{3}{1}$, so $a_2 = 3$ and $\{x_3\} = 0$.

Therefore, $x = \frac{24}{7} = [3; 2, 3]$.

Definition 2.13. (Burton, 1992) An *infinite continued fraction* is an expression of the following form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

where $a_{0,}a_1, a_2, a_3, \ldots$ are real numbers with $a_1, a_2, \ldots, a_n$ positive and $a_0 \geq 0$ and it is denoted by $[a_0; a_1, a_2, \ldots, a_n, \ldots]$. If the real numbers $a_0, a_1, a_2, \ldots, a_n$ are all integers, then the continued fraction is called *simple*.

Theorem 2.14. (Rosen, 1992) Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers with $a_1, a_2, \ldots$ positive, and let $C_k = [a_0; a_1, a_2, \ldots, a_k]$. Then, the convergents C_k tend to a limit α, i.e. $\lim_{k\to\infty} C_k = \alpha$.

Definition 2.15. (Stein, 2008) *A periodic continued fraction* is a continued fraction of the form $[a_0; a_1, a_2, \ldots, a_n, \ldots]$ such that $a_n = a_{n+t}$ for some fixed positive integer t and all sufficiently large n. Such a minimal t is called as the *period of the continued fraction*.

If the continued fraction contains no initial non-periodic terms, then it is called *purely periodic*.

Theorem 2.16. (Koshy, 2007) Let $\alpha = x_0$ be an irrational number. Define the sequence $\{a_k\}_{k=0}^{\infty}$ of integers a_k recursively as follows:

$$a_k = \lfloor x_k \rfloor, \qquad x_{k+1} = \frac{1}{x_k - a_k}$$

where $k \geq 0$. Then $\alpha = [a_0; a_1, a_2, \ldots]$.

Continued fraction expansion can also be found in the above form if α is an irrational number. Let's show this on an example.

Example 2.17. Express $\alpha = \sqrt{19}$ as an infinite simple continued fraction.

$$a_0 = \lfloor x_0 \rfloor = \lfloor \sqrt{19} \rfloor = 4, \qquad x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{19} - 4} = \frac{\sqrt{19} + 4}{3}$$

$$a_1 = \lfloor x_1 \rfloor = 2, \qquad x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{19} + 4}{3} - 2} = \frac{\sqrt{19} + 2}{5}$$

$$a_2 = \lfloor x_2 \rfloor = 1, \qquad x_3 = \frac{1}{x_2 - a_2} = \frac{1}{\frac{\sqrt{19} + 2}{5} - 1} = \frac{\sqrt{19} + 3}{2}$$

$$a_3 = \lfloor x_3 \rfloor = 3, \qquad x_4 = \frac{1}{x_3 - a_3} = \frac{1}{\frac{\sqrt{19} + 3}{2} - 3} = \frac{\sqrt{19} + 3}{5}$$

$$a_4 = \lfloor x_4 \rfloor = 1, \qquad x_5 = \frac{1}{x_4 - a_4} = \frac{1}{\frac{\sqrt{19} + 3}{5} - 1} = \frac{\sqrt{19} + 2}{3}$$

$a_5 = \lfloor x_5 \rfloor = 2$, $\quad x_6 = \dfrac{1}{x_5 - a_5} = \dfrac{1}{\dfrac{\sqrt{19}+2}{3} - 2} = \sqrt{19} + 4$

$a_6 = \lfloor x_6 \rfloor = 8$, $\quad x_7 = \dfrac{1}{x_6 - a_6} = \dfrac{1}{\sqrt{19}+4-8} = \dfrac{\sqrt{19}+4}{3} = x_1$

As it can be seen that $x_7 = x_1$. So, the pattern continues. Thus,

$\sqrt{19} = [4;2,1,3,1,2,8,2,1,3,1,2,8,\ldots] = [4;\overline{2,1,3,1,2,8}]$.

As can be seen, every irrational number can be represented by an infinite simple continued fraction (Koshy, 2007; Robbins, 1993).

Example 2.18. Express the purely periodic continued fraction $\alpha = [\overline{2;1}]$ in the form $a + b\sqrt{d}$, where $a, b \in \mathbb{Q}$ and d is a square-free integer greater than 1.

$$[\overline{2;1}] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

Since $\alpha = [\overline{2;1}]$, it can be written

$$\alpha = 2 + \cfrac{1}{1 + \cfrac{1}{\alpha}} = \frac{3\alpha + 2}{\alpha + 1}.$$

That is, $\alpha^2 - 2\alpha - 2 = 0$, so $\alpha = 1 + \sqrt{3}$.

Every purely periodic continued fraction is an infinite continued fraction. As can be seen from the example, the value of an infinite continued fraction is an irrational number. This explains the following theorem.

Theorem 2.19. (Burton, 1992) The value of any infinite continued fraction is an irrational number.

Theorem 2.20. (Mollin, 2008) If $C_k = \dfrac{p_k}{q_k}$, for $k \in \mathbb{N}$, is the *kth convergent* of an irrational number α, then the following holds

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.$$

For example, if $\sqrt{41} = [6;\overline{2,2,12}]$, it is obvious that $C_5 = \dfrac{p_5}{q_5} = \dfrac{2049}{320}$. So, $\left| \sqrt{41} - C_5 \right| < \dfrac{1}{q_5^2}$, where C_5 is the 5*th convergent* in the infinite continued fraction representation of $\sqrt{41}$.

3. Pell's Equation

In the literature, there are different methods for solving Pell's equation such as the Lagrange-Matthews-Mollin algorithm, the cyclic method, Lagrange's system of reductions, use of binary quadratic forms, etc. Here it will be explained how to solve Pell's equation using continued fractions.

Definition 3.1. (Robbins, 1993) The quadratic Diophantine equation of the form $x^2 - Dy^2 = N$, where D is a positive non-square integer and N is a non-zero integer, unknowns x and y are positive integers, is called as *generalized Pell's equation*. If $N = 1$, i.e. $x^2 - Dy^2 = 1$, then the form is called as *Pell's equation*. If $N = -1$, then the equation $x^2 - Dy^2 = -1$ is called as *associated Pell's equation (negative Pell's equation)*.

The *trivial solution* of Pell equation is $x = 1$, $y = 0$. There is a minimal solution to $x^2 - Dy^2 = \pm 1$ in positive integers which is greater than 1 is called as the *fundamental solution* of this equation.

Theorem 3.2. (Burton, 2011) If p, q is a positive solution of $x^2 - Dy^2 = 1$, then $\frac{p}{q}$ is a convergent of the continued fraction expansion of $\sqrt{D}$.

Theorem 3.3. (Rosen, 1992) Let (x_1, y_1) be the fundamental solution of the equation $x^2 - Dy^2 = 1$, where D is a positive integer that is not a perfect square. Then, all positive solutions (x_n, y_n) are given by

$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n$$

for $n = 1, 2, 3, \ldots$.

The next theorem will present several important tools for solving the Pell's equation.

Theorem 3.4. (Robbins, 1993) Let D be a positive non-square integer. Let t be the length of the period of the continued fraction expansion of $\sqrt{D}$. Then, Pell's equation $x^2 - Dy^2 = 1$ has infinitely many solutions, all are given as follows:

- **a.** If t is even, then $x_n = p_{nt-1}$, $y_n = q_{nt-1}$ for $n = 0, 1, 2, 3, \ldots$
- **b.** If t is odd, then $x_n = p_{2nt-1}$, $y_n = q_{2nt-1}$ for $n = 0, 1, 2, 3, \ldots$.

If the negative Pell equation $x^2 - Dy^2 = -1$ is examined, the following holds:

- **c.** If t is even, then the equation $x^2 - Dy^2 = -1$ has no solutions.
- **d.** If t is odd, then the equation $x^2 - Dy^2 = -1$ has infinitely many solutions, all given by $x_n = p_{nt-1}$, $y_n = q_{nt-1}$, where $n = 1, 3, 5, \ldots$.

Theorem 3.5. (Andreescu & Andrica, 2015) Let $p \geq 3$ be a prime. The negative Pell's equation $x^2 - Dy^2 = -1$ is solvable in positive integers if and only if $p \equiv 1 \pmod 4$.

Example 3.6. Solve the Pell's equation $x^2 - 63y^2 = 1$.

The solution depends on the continued fraction expansion of $\sqrt{63}$.

$$a_0 = \lfloor x_0 \rfloor = \lfloor \sqrt{63} \rfloor = 7, \qquad x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{63} - 7} = \frac{\sqrt{63} + 7}{14}$$

$$a_1 = \lfloor x_1 \rfloor = 1, \qquad x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{63} + 7}{14} - 1} = \sqrt{63} + 7$$

$$a_2 = \lfloor x_2 \rfloor = 14, \qquad x_3 = \frac{1}{x_2 - a_2} = \frac{1}{\sqrt{63} + 7 - 14} = \frac{\sqrt{63} + 7}{14} = x_1$$

As it can be seen that $x_3 = x_1$. So, the pattern continues. Thus,

$\sqrt{63} = [7;1,14,1,14,1,14,\ldots] = [7;\overline{1,14}]$.

The period length of the continued fraction $\sqrt{63}$ is 2, that is, even. $(p_1, q_1) = (8,1)$ i.e.

fundamental solution of the Pell's Equation $x^2 - 63y^2 = 1$ is $(x_1, y_1) = (8,1)$.

All positive integer solutions of the Pell's Equation $x^2 - 63y^2 = 1$ are given by

$$x_n + y_n\sqrt{D} = \left(8 + \sqrt{63}\right)^n$$

for $n = 1, 2, 3, \ldots$.

Example 3.7. Solve the Pell's equation $x^2 - 98y^2 = 1$.

The solution depends on the continued fraction expansion of $\sqrt{98}$.

$$a_0 = \lfloor x_0 \rfloor = \lfloor \sqrt{98} \rfloor = 9, \qquad x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{98} - 9} = \frac{\sqrt{98} + 9}{17}$$

$$a_1 = \lfloor x_1 \rfloor = 1, \qquad x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{98} + 9}{17} - 1} = \frac{\sqrt{98} + 8}{2}$$

$$a_2 = \lfloor x_2 \rfloor = 8, \qquad x_3 = \frac{1}{x_2 - a_2} = \frac{1}{\frac{\sqrt{98} + 8}{2} - 8} = \frac{\sqrt{98} + 8}{17}$$

$$a_3 = \lfloor x_3 \rfloor = 1, \qquad x_4 = \frac{1}{x_3 - a_3} = \frac{1}{\frac{\sqrt{98} + 8}{17} - 1} = \sqrt{98} + 9$$

$$a_4 = \lfloor x_4 \rfloor = 18, \qquad x_5 = \frac{1}{x_4 - a_4} = \frac{1}{\sqrt{98} + 9 - 18} = \frac{\sqrt{98} + 9}{17} = x_1$$

As it can be seen that $x_5 = x_1$. So, the pattern continues. Therefore,

$\sqrt{98} = [9;1,8,1,18,1,8,1,18,\ldots] = [9;\overline{1,8,1,18}]$.

The period length of the continued fraction $\sqrt{98}$ is $t = 4$, that is, even. $(p_3, q_3) = (99,10)$

i.e. fundamental solution of the Pell's Equation $x^2 - 98y^2 = 1$ is $(x_1, y_1) = (99,10)$.

All positive integer solutions of the Pell's Equation $x^2 - 98y^2 = 1$ are given by

$$x_n + y_n\sqrt{D} = \left(99 + 10\sqrt{98}\right)^n$$

for $n = 1,2,3,\ldots$.

Let's continue with the more general Pell's equations.

Application 3.8. Let $a \in \mathbb{N}$. Solve the Pell's equation $x^2 - \left(a^2+1\right)y^2 = 1$ for $a \geq 1$.

Continued fractions crop up a way to analyze solutions to Pell's equation. In this equation, the solution depends on the continued fraction expansion of $\sqrt{a^2+1}$. It can be seen that the continued fraction expansion of $\sqrt{a^2+1}$ is $\left[a;\overline{2a}\right]$ (Robbins, 1993, p. 225).

Let's find the continued fraction expansion ourselves.

$$\begin{aligned}\sqrt{a^2+1} &= a + \left(\sqrt{a^2+1} - a\right)\\ &= a + \frac{1}{\sqrt{a^2+1}+a}\\ &= a + \frac{1}{2a + \left(\sqrt{a^2+1} - a\right)}.\end{aligned}$$

Therefore, $\sqrt{a^2+1}$ has continued fraction representation $\sqrt{a^2+1} = \left[a;\overline{2a}\right]$. So, $t = 1$.

That is, the length of the period of the continued fraction expansion of $\sqrt{a^2+1}$ is odd.

So $(x_1, y_1) = \dfrac{p_1}{q_1} = \dfrac{2a^2+1}{2a}$ is the fundamental solution to the Pell's equation

$x^2 - \left(a^2+1\right)y^2 = 1$. Moreover, the positive solution set S to Pell's equation is as follows:

$$S = \left\{(x_k, y_k) \in \mathbb{Z}^2 : \left(x_k + y_k\sqrt{a^2+1}\right) = \left(2a^2+1+2a\sqrt{a^2+1}\right)^k\right\}_{k=1}^{\infty}.$$

Application 3.9. Let $a \in \mathbb{N}$. Solve the Pell's equation $x^2 - \left(a^2+2\right)y^2 = 1$.

Since all small values of $x^2-(a^2+2)y^2$ arise from convergents, if $x^2-(a^2+2)y^2=1$ is to have a solution it must arise from a convergent to $\sqrt{a^2+2}$. Therefore, it should be found a continued fraction representation of $\sqrt{a^2+2}$. It can be seen that the continued fraction expansion of $\sqrt{a^2+2}$ is $\left[a;\overline{a,2a}\right]$ (Robbins, 1993, p. 225).

Let's find the continued fraction expansion ourselves.

$$\sqrt{a^2+2}=a+\left(\sqrt{a^2+2}-a\right)$$

$$=a+\cfrac{1}{\cfrac{\sqrt{a^2+2}+a}{2}}$$

$$=a+\cfrac{1}{a+\cfrac{\sqrt{a^2+2}-a}{2}}$$

$$=a+\cfrac{1}{a+\cfrac{1}{\sqrt{a^2+2}+a}}$$

$$=a+\cfrac{1}{a+\cfrac{1}{2a+\left(\sqrt{a^2+2}-a\right)}}$$

Therefore, the continued fraction expansion of $\sqrt{a^2+2}$ is $\left[a;\overline{a,2a}\right]$. So, the length of the period of the continued fraction expansion of $\sqrt{a^2+2}$ is $t=2$. Then, the fundamental solution of the equation $x^2-(a^2+2)y^2=1$ is $(x_1,y_1)=\frac{p_1}{q_1}=\frac{a^2+1}{a}$. Moreover, the positive solution set S to Pell's equation is

$$S=\left\{(x_k,y_k)\in\mathbb{Z}^2:\left(x_k+y_k\sqrt{a^2+2}\right)=\left(a^2+1+a\sqrt{a^2+2}\right)^k\right\}_{k=1}^{\infty}.$$

Application 3.10. Let $a\in\mathbb{N}$. Solve the equations $x^2-(9a^2+3)y^2=1$ and $x^2-(9a^2+3)y^2=-1$.

Continued fractions provide a way to analyze solutions to the Pell's equation $x^2-(9a^2+3)y^2=1$. In this equation, the solution depends on the continued fraction expansion of $\sqrt{9a^2+3}$. It can be seen that the continued fraction expansion of $\sqrt{9a^2+3}$ is $\left[3a;\overline{2a,6a}\right]$ (Robbins, 1993, p. 226).

Let's find the continued fraction expansion ourselves.

$$\sqrt{9a^2+3}=3a+\left(\sqrt{9a^2+3}-3a\right)$$

$$= 3a + \cfrac{1}{\cfrac{\sqrt{9a^2+3}+3a}{3}}$$

$$= 3a + \cfrac{1}{2a + \cfrac{\sqrt{9a^2+3}-3a}{3}}$$

$$= 3a + \cfrac{1}{2a + \cfrac{1}{\sqrt{9a^2+3}+3a}}$$

$$= 3a + \cfrac{1}{2a + \cfrac{1}{6a + \left(\sqrt{9a^2+3}-3a\right)}}.$$

Therefore, $\sqrt{9a^2+3} = \left[3a; \overline{2a, 6a}\right]$. So, $t = 2$. That is, the length of the period of the continued fraction expansion of $\sqrt{9a^2+3}$ is even. So $(x_1, y_1) = \frac{p_1}{q_1} = \frac{6a^2+1}{2a}$ is the fundamental solution to the Pell's equation $x^2 - \left(9a^2+3\right)y^2 = 1$. Moreover, the positive solution set S to Pell's equation is as follows:

$$S = \left\{ \left(x_k, y_k\right) \in \mathbb{Z}^2 : \left(x_k + y_k\sqrt{9a^2+3}\right) = \left(6a^2+1+2a\sqrt{9a^2+3}\right)^k \right\}_{k=1}^{\infty}.$$

The length of the period of the continued fraction expansion of $\sqrt{9a^2+3}$ is even. Therefore, the equation $x^2 - \left(9a^2+3\right)y^2 = -1$ has no solutions.

Solutions of different Pell's equations can be found in the literature. For example, Peker and Senay (2015), found continued fraction expansion of $\sqrt{D}$ when $D = a^2 + 2a$ where a is positive integer. They solved the Pell's equation $x^2 - \left(a^2+2a\right)y^2 = N$ when $N \in \{\pm1, \pm4\}$ and they formulated n th solution via the generalized Fibonacci and Lucas sequences. Keskin and Güney Duman (2019), considered continued fraction expansion of $\sqrt{D}$ when $D = k^2 \pm 4$ and $D = k^2 \pm 1$. They solved the Pell's equation $x^2 - Dy^2 = N$ when $N \in \{\pm1, \pm4\}$. Raza and Malik (2018) extended all the results of the various papers about the Pell's equation $x^2 - Dy^2 = N$ when $N \in \{\pm1, \pm4\}$.

4. Conclusion

It has been presented a brief introduction to the theory of continued fractions. Continued fraction types are mentioned. Continued fractions have been introduced and have been applied these properties to solve Pell's equation. The Continued Fraction Algorithm is explained. It is stated that the continued fraction algorithm finishes after finitely many steps if and only if x is rational.

It was mentioned that how to determine rational and irrational numbers using continued fraction representations. Every nonzero rational number can be represented by a finite simple continued fraction and the value of any finite simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ is always a rational number. Every irrational number can be represented by an infinite simple continued fraction and the value of any infinite simple continued fraction $[a_0; a_1, a_2, \ldots, a_n, \ldots]$ represents an irrational number.

The quadratic Diophantine equation of the form $x^2 - Dy^2 = 1$, where D is a positive integer which is not a perfect square, unknowns x and y are positive integers, is called *Pell's equation*. In the literature, there are different methods for solving Pell's equation. Here, it was explained how to solve Pell's equation using continued fractions. The solution depends on the continued fraction expansion of $\sqrt{D}$. The fundamental solution of Pell's equation is found by using convergents of $\sqrt{D}$. All solutions of Pell's equation can be reached using the fundamental solution. The complete set of solutions to Pell's equation is the infinite cyclic group generated by the fundamental solution.

Finally, applications have been made on various Pell's equations by determining continued fractions for square roots of positive integers initially and after that applying their results to solve Pell's equation.

5. References

Andreescu, T., Andrica, D., & Cucurezeanu, I. (2010). *An introduction to Diophantine equations, a problem-based approach*, Springer Science+Business Media.

Andreescu, T., & Andrica, D. (2015). *Quadratic Diophantine equations*, Springer Science+Business Media.

Burton, D. M. (2011). *Elementary number theory* (7th Edition), McGraw-Hill.

Coppel, W. A. (2006). *Number theory-an introduction to mathematics: part A*, Springer Science+Business Media.

Hardy, G. H. & Wright, E. M. (2008). *An introduction to the theory of numbers* (6th Edition), Oxford University Press Inc.

Keskin, R., & Güney Duman, M. (2019). Positive integer solutions of some Pell equations, *Palestine Journal of Mathematics, 8*(2), 213-226.

Koshy, T. (2007). *Elementary number theory with applications* (2nd Edition), Elsevier.

Long, C. T. (1987). *Elementary introduction to number theory*, Prentice-Hall.

Mollin, R. A. (2008). *Fundamental number theory with applications* (2nd Edition), Chapman & Hall/CRC.

Olds, C. D. (1963). *Continued fractions*, Random House.

Peker, B., & Senay, H. (2015). Solutions of the Pell equation $x^2-(a^2+2a)y^2=N$ via generalized Fibonacci and Lucas numbers, *Journal of Computational Analysis and Applications*, *18*(4), 721-726.

Raza, Z., & Malik, H. M. (2018). Solution of certain Pell equations, *Asian-European Journal of Mathematics, 11*(2), 1850056 (17 pages).

Robbins, N. (1993). *Beginning number theory*, William C. Brown Pub.

Rosen, K. H. (1992). *Elementary number theory and its applications* (3rd Edition), Addison- Wesley Publishing Company.

Silverman, J. H. (2013). *A friendly introduction to number theory* (4th Edition), Pearson.

Stein, W. (2008). *Elementary number theory: primes, congruences, and secret: a computational approach*, Springer Science+Business Media.

About The Author

Bilge PEKER received her PhD degree in Mathematics from Selcuk University. Currently, she is working as an Associate Professor of Mathematics in the department of mathematics and science education at Necmettin Erbakan University where she is conducting research activities not only in the areas of algebra and number theory, especially Diophantine equations, Pell equations, Diophantine m-tuples but also in areas of mathematics education.

E-mail: bpeker@erbakan.edu.tr, ORCID: 0000-0002-0787-4996.

To Cite This Chapter

Peker, B. (2021). Continued fractions and Pell's equation. In M. Ozaslan & Y. Junejo (Eds.), *Current Studies in Basic Sciences, Engineering and Technology 2021*(pp. 127–142). ISRES Publishing