Conceptual Understanding and Procedural Knowledge of Fractions: How to Learn and How to Teach

Vyomesh Pant
Ministry of Finance, India

Mathematics is commonly considered as one of the most difficult subjects to study. 'Math Phobia' or 'Math Anxiety' is very common among the people, especially among the children. Due to intricacies involved in Mathematics, most of the children are either not interested or least interested to learn it. Inside mathematics, there are certain topics wherein the students, for a variety of reasons, often face difficulty in understanding and in conceptualizing them. Decimal, Fractions and Percentage are among these topics, where not only the students struggle to understand the concepts but the teachers also find it very tedious to teach these phenomena to their students. Researchers have corroborated the aspect that the students face difficulties in developing conceptual understanding of fractions and decimals (see Condon & Hilton, 1999). Many tests and research studies have documented that understanding of fractions is week among the students of every grade ranging from Primary to Secondary and even to higher grades also (see Armstrong & Larson, 1995; Erlwanger, 1973; Empson & Levi, 2011; Kamii & Clark, 1995; Mack, 1990; Moss & Case, 1999; Orpwood, Schollen, Leek, Marinelli-Henriques & Assiri, 2011; Post, Cramer, Lesh, Harel, & Behr, 1993; Perle, Moran & Lutkus, 2005; Stigler, Givvin & Thompson, 2010). Various suggestions have been made by the researchers to improve the learning of fractions. However, the teaching and understanding of fraction still poses significant challenge.

The paper is intended to find out the reasons which make the aspect of 'fraction' more difficult for the teachers to teach and for the students to understand. The paper also suggests the teaching methodology which may be adopted to make easy the understanding and conceptualization of fractions. A model has been presented to introduce the term 'fraction' and preliminary mathematics associated with it to the beginners in a way so that the children can understand and conceptualize fractions easily in a natural way. This model may help to overcome the difficulties which come while teaching and learning fractions. We also provide answers to the problems posed by Siegler et al (see Siegler, Fazio, Bailey & Zhou, 2013). We start with analysing the problems which come during the teaching of fractions and associated reasons. The present study may be seen as interdisciplinary work which associates the studies in Mathematics Education with that in Psychology, Medical Science and Cognitive Science.

Difficulties in Teaching and Learning Fractions

The main reasons which create difficulty in learning and teaching of fractions have been discussed in this Section.

Multiple forms of fractions

Fractions can be expressed in multiple forms. Accordingly, these can be introduced to the children in different ways, which are as follows:

1. A fraction in bipartite form, i.e., in the form of x/y, where x and y are whole numbers and y ≠ 0.
2. x/y can be expressed as 'x divided by y' or x ÷y,.
3. x/y also denotes the ratio of two numbers x and y and can also be represented as x:y.
4. x/y can be expressed as 'x out of y', more precisely, x/y can be expressed as 'x number of parts out of total y number of equal parts of the whole.
5. x/y is also used as an operator. For e.g., when we say 3/4 of 100, it means (3/4) x 100 = 75. In that case 3/4 is an operator, which when multiplied by a number, reduces to the said number to its 'three fourth' value.

Kieren had identified five different interpretations (or sub-constructs) of rational numbers (see Kieren, 1980). These are often summarized as part-whole, measure, quotient or division, operator and ratio. Due to variety of forms, it is a challenge for a teacher to choose the appropriate form to introduce fractions to the kids and subsequently expose them to all the forms one by one without creating any doubt or confusion in their minds. The basic difficulty before a teacher is that how to introduce fraction – in bipartite form or as a ratio or as division of two numbers or as an operator. On the other hand, the children get confused due to the abstract nature of the 'fractions' and with various forms it is used in. While learning fractions, much of the confusion among the students occurs because they are either not familiar with a variety of representations or they cannot synthesize different interpretations or sub constructs (see, Clarke, Roche & Mitchell, 2007). Most of the students are not able to develop understanding of fractions due to the multiple forms it is introduced in and as a result they start focusing on surface similarities of the representations rather than numerical meaning and understanding of fractions. (see Moseley & Okamoto, 2008; Clarke, Roche & Mitchell (2007; 2011)).

The students generally fail to perceive the fractions like numbers (Park, Flores & Hohensee, 2016; Hannula, 2003; Post, et al, 1993). Instead they visualize fractions as some separate entity (in form of or) made up from numbers and perceivethe fractions as an object disconnected from the number system.

Absence of natural reference point

According to Wu (see Wu, 2014), the reasons which make difficult the learning of fraction are absence of a natural reference point and the inherent abstract nature of the concept of fraction. To deal with the learning of whole numbers in grades 1-4, children always have a natural reference point : their fingers. The modeling of whole numbers on one's finger is both powerful and accurate (see Wu, 2014). In case of fractions, it is normally introduced using a pizza or pie or bread and thus the pizza, pie or bread become the reference points. These may be good reference points to help the beginners for the purpose of the vocabulary learning aspect. However, these models become awkward in case of fractions bigger than 1 or in describing mathematical operations like addition, subtraction, multiplication etc. For e.g., one cannot multiply two pieces of a pie (Wu, 2014; Hart, 2000).

Presentation/ interpretation of fractions is complex for the beginners

A fraction comprises of three parts – a numerator, a denominator and a line which separates the numerator and denominator. Normally, it is taught that the numerator represents a counting number and signifies "how many", whereas the denominator is named as the ordinal number which signifies "what is being counted" (see Watanabe, 2002). Both the numerator and denominator are whole numbers (with a restriction that denominator is always non-zero), but their behavior in the operations related to fractions remain different. For e.g., while adding two fractions, LCM of the numbers in denominator is taken and equivalent fractions are formed. During the addition the denominator remains as it is (equal to LCM), whereas the numbers in numerator are simply added as whole numbers. Therefore, for the children, particularly for those who are in the early stage of learning, numerator and denominator become two separate entities and the bar separating them is a third one, which makes the things difficult. A considerable amount of attention of the kids gets utilized to understand these three entities and a less amount is left to understand the structure and operations related to fractions and its execution. The greater memory load of representing fractions reduces the cognitive resources available for thinking about the procedure needed to solve the problems related with fractions (see Lortie-Forgues, et al, 2015; Fuchs, et al, 2013; 2014; Hecht & Vagi, 2012; Jordan et al, 2013; Siegler & Pyke, 2013).

Fractions do not behave like whole numbers

One of the main difficulties while learning fractions is that often learners use the natural number properties to make inferences on rational numbers, which was termed as "whole number bias" by Ni & Zhou (2005). It is a general belief among the learners that the properties of fractions (rational numbers) would be same as that of the whole numbers (see Siegler et al, 2011). The rational numbers form a densely ordered set, whereas the whole numbers are part of a discrete set. As such the rational numbers (fractions) possess certain properties which are different from corresponding properties of the whole numbers. However, for the beginners it often becomes difficult to adapt these difference and they expect the rational numbers (fractions) to behave like whole numbers. For many people the fraction consists of two whole numbers a and b and with this understanding they process numerator and denominator as two separate whole numbers (see Vamvakoussi & Vosniadou, 2004). There are some questions related to the basic arithmetic operations on fractions, which creates doubts in the minds of the learners. For e.g., while adding or subtracting the fractions, it is necessary to form equivalent fractions in such a way that denominator of all the fractions become equal (equal to the LCM), whereas while multiplying or dividing the fractions, there is no such requirement of making equivalent fractions. Further, in case of addition or subtraction, after making the equivalent fractions, only numerators are added or subtracted but denominators are not added or subtracted. Contrary to the practice followed in fraction addition or subtraction, both numerator and denominator are multiplied in case of fraction multiplication. In case of fraction division, the denominator is reversed and then multiplied with numerator. Though all these queries can be answered mathematically, but these are not normal to understand, does not appear natural and are far from obvious. From the point of view of a beginner, such problems arising during the basic fraction arithmetic are away from realization, answers to these problems are not apparent and this is one of the important reasons which create difficulty in understanding fractions.

Conceptual understanding and procedural knowledge

Studies have shown that while learning fractions, students perform calculations without understanding mathematics behind it (see Kerslake, 1986, Gabriel et al, 2013). The thrust of the teachers generally remains in imparting procedural knowledge and, less, or, no emphasis is given to conceptual understanding of fractions. Children usually learn rote procedures, which lead to misunderstanding (see Byrnes & Wasik, 1991; Gabriel et al, 2013). Conceptual knowledge is a must to have explicit or implicit understanding of the principles, central concepts, their interrelations and further generalizations (Rittle-Johnson & Alibali, 1999; Schneider & Stern, 2005; Hiebert, 1986; Gabriel et al, 2013) whereas the procedural knowledge is about the sequences of actions useful to solve the problems (Rittle-Johnson & Alibali, 1999). The conceptual and procedural knowledge are complementary to each other and both of them must co-exist in an iterative and interactive way to make the learning thorough, interesting and purposive (Rittle-Johnson & Alibali, 1999).

Lortie-Forgues et al have presented a fine survey of the inherent and culturally contingent difficulties comes in the way of learning fractions and decimals (see Lortie-Forgues, et al, 2015). Magnitude plays an important role in arithmetic of whole numbers. When we talk of a whole number, say 7, it means that its magnitude is 7. While thinking of a number, it is the magnitude which gives us impression about its position or place in the number system. In case of whole numbers or integers, it is quite easy to find out the magnitude of any number and, as such, to imagine position of the number in the number system. Two whole numbers or integers can be compared to find out which one is greater. Same degree of comparative ease is available in addition, subtraction and multiplication of the whole numbers. The operation of division in whole numbers or integers creates more problem as compared to the others. The division is not always quick and this is the reason due to which it is not easy to compare two fractions having different denominators. It is not easy to find out the magnitude of a fraction, which creates problem for the learner to determine the position or place of a fraction. Comparing two whole numbers is easy task but comparing two fractions, particularly those having different denominators is not always easy. Unlike whole number magnitudes, fraction magnitudes have to be derived from the ratio of the two values, which reduces the accuracy, speed and automaticity of access to the magnitude representations (English & Halford, 1995). Accessing fraction magnitude also requires understanding whole number division, often considered the hardest of the four arithmetic operations (Foley & Cawley, 2003).

Other factors

In addition to this, there are some factors which are also responsible for difficulties faced by the learners. Quality of teaching and the knowledge of the teacher play pivotal role in minimizing the difficulties of the learners. The social, cultural, educational and financial background of the learners as well as of the teacher also make a great impact on the learning and understanding of the children.

How to Overcome the Problems

Throughout this paper, when we form fractions using integers, we talk of non-negative integers only. Thus the term rational number for expressing the fractions has been used in the sense of non-negative rational numbers only.

In the study of fractions, the most important part is introduction, i.e., the way the fractions are introduced to the beginners. The way how the fractions are introduced is very crucial for the beginners. As explained in the previous paragraphs, there are number of ways to introduce fraction and a child must get himself familiarised with all these expressions gradually in due course of learning. However, if all the forms are taught simultaneously, it may create confusion. Instead, we must go on one by one and initiate in such a way that the pupils may not feel the burden of being taught an entirely new concept.

Fractions may be introduced as a part of the number system

Numbers occupy an important place in mathematics. Infact, our primary exposure with mathematics starts with numbers and during initial few years of learning, mathematics is all about the numbers only. A child is initially exposed to numbers, counting, reverse counting, number line, comparison of numbers and basic algebraic operations on numbers. During initial phase of learning mathematics, numbers signify the whole numbers only. Gradually, with increase in the level of learning, natural numbers, integers, rational and irrational numbers are introduced as a part of the framework of the number system.

It may be better for the beginners if rational numbers are taught before introducing the term 'fraction'. The following steps represent one of the models of teaching to introduce fraction :

Step 1: After having understood the concept of number line, the students may be asked to imagine in the number line, "what will be there between 0 and 1?", "what is between 4 and 5?", or in general, "What is between any two consecutive whole numbers?". Simultaneously, sharing problems may also be given to the children. For e.g., they may be asked to share 20 candies among 4 students, in which each of them will get 5 candies. If the number of candies is increased to 21, 22 or 23, it is not possible for them to share whole number of candies equally. They will be able to share equally, if the number of candies is increased to 24.In that case each of them will get 6 candies. Such type of examples may help the children to understand the requirement of numbers which are different from the whole numbers and may fill up the gaps between the whole numbers or integers. This forms basis for introduction of rational numbers. Each point of the Number Line represents a number, as such, there exist infinite numbers in the number line, all of them are not whole numbers or integers. Initially, there may be no need to classify these numbers into rational and irrational numbers. Infact, the irrational numbers may be kept at abeyance for quite some time because it may create some difficulty for the children at the initial stage, but the numbers in the form a/b : a and b are non-negative integers and b ≠ 0, may be taught and the children may visualize position of these numbers in the number line. Initially there is no need to use the term 'fraction', instead, the terminology of 'rational number' would be sufficient to understand the term and mathematics behind it. Further, the teacher must confine him to the non-negative integers only. In this way fractions may be introduced as rational numbers, as a part of number system, which will not only strengthen the concept number system but also familiarise childrenwith the essence of fractions, without burdening their minds with the load of the new terminology (fraction). The learners will be able to understand the rational numbers in a natural way as an extension of the number system. Further, having less memory load of introducing fractions will certainly increase the cognitive resources available for thinking about the structure and mathematical operations associated with fractions.

Step 2: Rational Numbers can be expressed as the numbers which are made of two integers a and b placed in the form a/b, b ≠ 0. Initially the students may be taught about positive rational numbers only. The integers and the whole numbers can also be expressed as rational numbers. For e.g., 2 can be expressed as 4/2 , 7 can be written as 7/1 or 14/2 and so on, which may help children in understanding that the rational numbers include the natural numbers, whole numbers and the integers. In the number line, the rational numbers can be represented by the dots of the whole numbers or integers and numerous other dots between these whole numbers or integers.

Step 3: The study of rational numbers is quite interesting and it can be made more interesting by highlighting the properties which are possessed by the rational numbers and not by the numbers the children are familiarise with, i.e., integers or whole numbers. The whole numbers as well as the integers form a discrete set, whereas the rational numbers form a densely ordered set. As such there are certain properties which differentiate the rational numbers from rest of the numbers. Some of these properties, which may be discussed to make the study of rational numbers more natural and purposeful, are as follows:

(i) Integers are not closed under the operation of division, but the rational numbers (excluding zero) are closed under the operation of division.

(ii) There are fixed or countable numbers of integers between any two integers. However, in case of rational numbers, there lies infinite number of rational numbers between any two integers or between any two rational numbers. For e.g., consider two rational numbers 3/10 and 7/10. There are 3 integers between 3 and 7, viz., 4, 5 and 6. Therefore, the corresponding rational numbers between 3/10 and 7/10 are 4/10 , 5/10 and 6/10. Now write 3/10 and 7/10 as 30/100 and 70/100 respectively and count the rational numbers between these two numbers, viz,31/100, 32/100, ...69/100. Again write 3/10 and 7/10 as 300/1000 and 700/1000 respectively and count the rational numbers between these two numbers. The process can go on and thus infinite numbers of rational numbers can be accommodated between any two rational numbers.

(iii) Between any two given numbers (integers or rational numbers), there may or may not exist an integer but it is for sure that there will always be a rational number between any two numbers. For e.g., there is no integer between 3 and 4 but there are infinite number of rational numbers between these numbers, such as, 7/2, 10/3, 13/4, ... etc.

(iv) In the set of integers, every integer can be represented by a unique symbol but a rational number can be represented by infinite number of expressions. For e.g., 1/2 = 2/4 = 3/6 = 4/8 = ... Though an integer can also be written in this manner, like 2 = 2/1, 4/2, 6/3 =..... , but we should not forget that for making such expressions we have to come out of the set of integers and use the set of rational numbers. Infact the integers are also rational numbers and, as such, hold the properties of rational numbers also. Inside the set of integers, it is not possible to write any rational number in multiple forms.

(v) In the set of integers, every integer has a unique predecessor and successor. However, in case of rational numbers, every rational number has infinite number of predecessors and successors.

(vi) Before learning rational numbers, the general perception of the children may be that the multiplication always increases the value of multiplicand as well as that of the multiplier. However, this is not true in case of integers. For e.g., 8 x (1/2) = 4 , which is less than 8.

(vii) Similarly, in case of division of rational number, the result may be greater than the dividend, which is contrary to the expectations of the learners because so far they are exposed to the division of integers only where the division always reduces the magnitude of the dividend. For e.g., (1⁄2)/(1⁄4)= 1/2 × 4/1=2 , which is greater than.

From the above, it is obvious that there are certain properties of rational numbers, which are entirely different from integers or whole numbers, due to which the rational numbers are required to be studied separately. The algebra of the rational numbers is also different from that of integers or whole numbers. Therefore, there is a need to study how the basic mathematical operations can be performed in case of rational numbers.

<u>Step 4</u>: Now the point to ponder is that despite so many differences, is there something which is common between the whole numbers and rational numbers. This insists the learners to identify the property which is common between the whole numbers and rational numbers. Obviously, the property that unites all the real numbers is that they possess magnitudes on the basis of which the numbers can be ordered in the number line. The magnitude of the fraction (rational numbers) plays crucial role in conceptual understanding and procedural knowledge of fractions. Once the children understand the magnitude of a rational number, they are able to visualize its position in the number line. The magnitude of a rational number and its position in the number line are the aspects which help the learner to understand that the rational numbers are part of the number system. Infact, these two aspects – magnitude and position in number line of

rational number – form the basis for conceptualization of rational numbers or fractions.

Advantage of introducing fraction in form of rational numbers

The study of fractions as rational number as an extension of the existing number system comes in a natural way for the beginners. Other sub-constructs are not so natural or self-evident during the initial phase of learning. The part-whole concept does not work when it comes to the fractions which are greater than 1 or the negative fractions. 'x out of y' type explanation of fractions creates problems for the learners, when improper fractions like 5/3 are to be understood – 'how can one take 5 parts out of total 3 parts?' If the fractions are introduced as an operator or decimal numbers, there are several other constraints of these methods, particularly from the point of view of the beginners. Fraction as an operator can reduce the value of any integer on multiplication and increase its value on division. This behaviour is quite opposite to that of the whole numbers and creates confusion. However, if factions are introduced as non-negative rational numbers in a natural way, as described above, it will become easy for the learners to understand the requirement of introducing rational numbers and conceptualize their existence and position in the number line.

After understanding the rational numbers, the learners become able to recognize that the number system is not confined to the whole numbers and there exists other type of numbers also. Further, there are many properties of the whole numbers which are not true for all the numbers, in general. Some of these properties have been highlighted in Step 3 above. Simultaneously, Step 4 provides the properties which are common for both the whole numbers and the rational numbers. Such comparison will make the study interesting and will help in conceptualization and understanding as well.

Neuroscience corroborates that there is underlying commonality in the neural basis of whole number and fraction knowledge

Studies in Neuroscience have revealed that the intraparietal sulcus (IPS), which is located in human brain on the lateral surface of the parietal lobe, plays a role in processing of numerical information. It has been found in the study of brain that while solving arithmetic problems brain activity shows strong commonalities between whole numbers and fractions (Schmithorst & Brown, 2004). An independent components analysis of brain activity during addition and subtraction of fractions, as measured by functional Magnetic Resonance Imaging (fMRI), revealed task-related components with activation in bilateral inferior parietal, left perisylvian, and ventral occipitotemporal areas, a pattern closely similar to that observed with whole number arithmetic (Dehaene et al, 2003). These results suggest an underlying commonality in the neural basis of whole number and fraction knowledge (see Siegler, Fazio, Bailey & Zhou, 2013). Studies have shown that the behavioural and neural methods which have proved useful for understanding whole number representations and processes are also useful for understanding representations and processes involving fractions (Siegler, Thompson, & Schneider, 2011; Ischebeck, Schocke & Delazer, 2009; Jacob, Vallentin & Nieder, 2012; Schneider & Siegler, 2010). It has been found that understanding of fractions is centered around understanding the magnitude of the fraction in a similar manner like the understanding of whole numbers is centered around understanding of their magnitudes (see Siegler, Fazio, Bailey & Zhou, 2013; Dehaene, 2011; Opfer & Siegler, 2012). The problem in conceptualization of the fractions is due to the fact that the learners often cannot visualize the magnitude of the fraction, and, accordingly, its position in the number system. The problem can be overcome by introducing the fractions as rational numbers specifying their magnitude and position in the number line.

It has been found in the neuroimaging studies that like integers, the fraction comparisons also exhibit distance effect, however, the size and scale of the effect is entirely different from the corresponding outcome in case of the integers and it depends upon the holistic magnitudes of the numbers being compared. The response time and error rate for fraction comparison were found much higher than that in case of integers. (DeWolf, Chiang, Bassok, Holyoak, & Monti, 2016; DeWolf, Grounds, Bassok, & Holyoak, 2014). Neuroimaging results suggest that the brain represents proportional (fraction) magnitudes in the same way that it does absolute (integer) magnitudes (Ischebeck, Schocke & Delazer, 2009; Jacob & Nieder, 2009). In a subsequent study, it is found that the brain processes the fractions in different manner as it does in case of integers (see DeWolf, Grounds, Bassok, & Holyoak, 2014). However, the reason behind the difference in processing integers and rational numbers may be attributed to the procedural and conceptual competence of the person on which the experiment was performed. If the fractions are understood as rational numbers and are conceived in ordered mental continuum as per their magnitudes, there might be no considerable difference between the integers and fractions (rational numbers). The fact which cannot be ignored here is that it is always not very easy to get the magnitude of a rational number, therefore, the response time may vary. The fraction magnitudes are much more difficult than the whole-number or integer magnitudes. Hurst & Cordes have found that in the study of fractions carried out by them, the adults fixated significantly longer at the numerator of the fractions relative to the denominator. Individual differences in this measure of whole-number bias suggested that looking more at the numerator than at the denominator implies poorer understanding of fraction procedures (see Hurst & Cordes, 2016).

If the fractions are introduced as rational numbers, as an extension of the number system, the learner will be able to understand the magnitude of the rational numbers and on the basis of their magnitude, they will also be able to determine its position in the number line. Further, in this form the learner will recognize the fraction as one entity and instead of focusing more on numerator, the learner will look at the complete fraction (rational number). The clinical results support the assertion that the fractions as rational numbers are not difficult to understand if they are taught properly.

Number line acts as a reference point

Study of rational numbers using the number line provides the learner a reference point, i.e., the number line. They may visualize the position of the rational numbers on it, which may help them in understanding the number system in better way. Further, the number line becomes a reference point which may help the students to visualize the rational numbers, which may addresses the concern posed by Wu (see Wu, 2014) and Hart (see Hart, 2000) that the difficulty in teaching and learning fraction is due to absence of a reference point.

Optimal use of cognitive resources

It may be recalled that much of the confusion in teaching and learning fractions is because fractions are introduced in various forms and the students cannot synthesize the different interpretations or sub-constructs of fractions (see Clarke, et al, 2011). Therefore, instead of teaching fractions in different forms, it would be better if the rational numbers are introduced first as an extension of number system, without referring to the term fraction. Initially there is no need to use the terminology 'fraction' or any other sub-construct of it to name fraction. If fractions are introduced as rational numbers, without using the term fraction, this may allow the beginners to pay their whole attention towards learning of rational numbers. While learning fractions, the greater memory load of representing fractions reduces the cognitive resources available for thinking about the procedure needed to solve the problems related with fractions (see Lortie-Forgues, et al, 2015; Fuchs, et al, 2013; 2014; Hecht & Vagi, 2012; Jordan, et al, 2013; Siegler & Pyke, 2013). To address this problem, it would be better if fractions are introduced as rational numbers so that all the cognitive resources available for thinking may be utilized in learning intricacies of rational numbers.

The pupil understand the algebra of fractions before formal introduction of fractions

Having expressed the pupils the properties of the rational number with illustrative examples, we can show that the rational numbers behave in different manner and unlike the whole numbers or integers. After introduction of rational numbers, beginners may be told about the addition, subtraction, multiplication and division of rational numbers. Remember that the term fraction has not been introduced to the learners and we have to focus on the algebraic properties of the rational numbers. After explaining the children the properties of the rational numbers, it becomes natural for them to adapt that the operations of addition, subtraction, multiplication and division of rational numbers will be different from that of the whole numbers or integers.

How to make the learning easy for the beginners

Use of geometry and multimedia can make the things easy

The Cognitive Load Theory (CLT), developed by John Sweller, is considered as one of the most influential theories in instructional design. It states that the instructional design can be used to reduce cognitive load among learners. CLT has been designed to provide guidelines intended to assist in the presentation of information in a manner which encourages learner activities that optimize intellectual performance (see Sweller, 1988; Sweller, et al, 1998). As per the CLT, there are three types of cognitive loads – Intrinsic Cognitive Load, which refers to the effort associated with a specific topic, Extraneous Cognitive Load, which describes the mode or the way of presenting the topic before a learner and Germane Cognitive Load, which indicates the efforts done to create a permanent store of knowledge in the learner. The method of teaching fractions step by step must be designed keeping in view the CLT. Another important theory is the Cognitive Theory of Multimedia Learning (CTML), which was popularized by the work of Richard E. Mayer who argues that multimedia supports the way that the human brain learns (Mayor, 2010; 2010). As per the theory of Mayer, the people learn more deeply from words and pictures than from words alone. Such kind of learning is referred to as the multimedia principle. Multimedia researchers generally define multimedia as the combination of text and pictures and suggest that multimedia learning occurs when we build mental representations from these words and pictures. The words can be spoken or written, and the pictures can be any form of graphical imagery including illustrations, photos, animation, or video. Multimedia instructional design attempts to use cognitive research to combine words and pictures in ways to optimize the learning effectiveness. Following the CLT and CTML, efforts may be made by the teachers to provide geometrical interpretation of every concept so that the children may be able to understand it properly and memorize it in better way. Stepanek et al have emphasized on the need of teaching elementary mathematics by Design (see Stepanek, et al, 2011). The term design generally refers to the creation of a product in an artistic or highly skilled manner. Good design means that the product fits the needs of the people who will use it and the context in which it will be used. Teaching by Design is a way to describe teachers' work that focuses on planning deliberate and purposeful lessons that fit the needs of their students.

Fraction knowledge may be linked with real world applications

All too often learners think of 'fractions' as being a discrete (and often difficult) topic that has no real connection with any other area of mathematics (see McLeod & Newmarch, 2006). However, if meaningful connections are made, the learning becomes more powerful. While attempting such connections, the teacher should be extra cautious that in the endevour to make it understandable to the pupils, the basic structure of the mathematical phenomenon of fraction is not distorted. Mathematics is often considered as 'abstract' or 'dry' subject by a large section of society. To make it understandable and interesting for all, various examples are created by the teachers so that all the student can attach them and learn it without any fear. Being abstract is an underlying essence of mathematics and one enjoys this flavor of mathematics only after proper conceptualization of any mathematical phenomenon. It is also true that Mathematics is often connected with real world applications to make it interesting and purposeful. Therefore, work on fractions needs to be integrated with other topics of mathematics; number, shape, data handling, and particularly every sort of measure of weight, length, capacity, time, and simple probability. Learners encounter fractions throughout their work at all entry levels of the numeracy curriculum, e.g., solving money problems, sharing a bill, comparing prices, calculating journey times, cooking, interpreting data in pictograms and bar charts, using a meter rule, measuring a room, comparing each other's heights, and checking the weight of ingredients (see McLeod & Newmarch, 2006), which may be connected with the learning of fractions.

Discussion

This paper suggests method to teach fractions as extension of number system, i.e., in form of rational numbers. The technique suggested in this paper is very simple, natural and keeps the children connected with the number system so that they can understand and conceptualize it. During further study, they may be taught the rational numbers are also known as fractions.

We also provide answers to the problems posed by Siegler et al (see Siegler, Fazio, Bailey & Zhou, 2013). Some of the problems posed by Siegler et al are as follows:

> "What relations connect whole number and fractions knowledge? Are individual differences in representations of whole number and fraction magnitudes related, do earlier individual differences in whole number representations predict later differences in fraction representations, and do interventions that improve whole number magnitude representations improve fraction representations as well?"

> "How specific are the relations of neural processing of whole numbers and fractions? For example, are individual differences in brain activations on the two types of numbers related? Do interventions that improve whole number and fraction magnitude representations produce similar changes in brain activity?"

The present paper provides answers to the above problems. Clearly it is the magnitude that relates the whole numbers and fractions. Number line acts as a frame of reference for all the numbers and the numbers are positioned in it in order of their magnitudes. Two things are required to conceptualize a number – position of the number in the number line (to understand existence of the number) and ordering of the number with reference to other numbers. It's the magnitude of the whole number which determines the position of the number in the number line and also decides its order as to which numbers are superseded by it and which one are succeeded. As described in this paper, if the fractions are introduced as rational numbers and emphasis is given to find out and understand their magnitudes, it will help the learners to determine position of the rational numbers in the number line, which, in turn, will help them to understand the position of the rational numbers and their ordering in the number line. Clear understanding of the position and ordering of the rational numbers, which is based on their magnitudes, will help the learners to conceptualize the rational numbers (fractions).

It has been found in the study of brain that while solving arithmetic problems brain activity shows strong commonalities between whole numbers and fractions (Schmithorst & Brown, 2004). The fMRI analysis of brain activity during addition and subtraction of fractions exhibits a pattern closely similar to that observed with whole number arithmetic (Dehaene et al, 2003). The problem in conceptualization of the fractions is due to the fact that the learners often cannot visualize the magnitude of the fraction, and, accordingly, its position in the number system. The problem can be overcome by introducing the fractions as rational numbers specifying their magnitude and position in the number line.

Neuroimaging results suggest that the brain represents proportional (fraction) magnitudes in the same way that it does absolute (integer) magnitudes (Ischebeck, Schocke & Delazer, 2009; Jacob & Nieder, 2009). If the fractions are understood as rational numbers and are conceived in ordered mental continuum as per their magnitudes, there might be no considerable difference between the integers and fractions (rational numbers). The fact which cannot be ignored here is that it is always not very easy to get the magnitude of a rational number, therefore, the response time may vary. The fraction magnitudes are much more difficult than the whole-number magnitudes and this facts give slight edge to the whole numbers. The brain activation in case of fractions may be different due to two factors – poor understanding of fractions and difficulty in calculation of their magnitudes as compared to the whole numbers. The difficulty in calculation of the magnitude of fractions is inherent or structural and will persist but the poor understanding of the fractions can be rectified by introducing them as rational numbers emphasising focus on their magnitudes. If the fractions are taught in the manner as suggested in this paper, the neural results will certainly exhibit better results and the change in brain activity in case of the whole numbers vis-à-vis the rational numbers will reduce to the minimum level leaving the part aside which may persist due to inherent structure of the fractions.

References

Armstrong, B.E. & Larson, C.N. (1995). Students' use of part-whole & direct comparison strategies for comparing partitioned rectangles, *Journal of Research in Mathematics Education*, 26(1), 2–19.

Bruce, C., Chang, D., & Flynn, T. (2013). *Foundations to Learning and Teaching Fractions: Addition and Subtraction – Literature Review*, Report submitted to Curriculum and Assessment Branch, Ontario Ministry of Education, ON K9J 0G2, Canada.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Dev. Psychol.* 27, 777–786. doi: 10.1037/0012-1649.27.5.777

Clarke, D., Roche, A. & Mitchell, A. (2007). Year six fraction understanding: A Part of the Whole Story. In J. Watson & K. Beswick (Eds), *Proceedings of the 30th Annual Conference of the Mathematics Education Group of Australasia*, pp. 207–216. Adelaide, Australia: MERGA Inc.

Clarke, D., Roche, A., & Mitchell, A. (2011). Year Six Fraction Understanding: A Part of Whole Story. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australia* (Vol. I, pp.207-216). Adelaide, Australia: MERGA Inc.

Condon, C., & Hilton, S. (1999). Decimal Dilemmas. *Australian Primary Mathematics Classroom*, 4(3), 26-31.

Dehaene, S. (2011). The Number Sense: How the Mind Creates Mathematics, In Revised and Updated Edition, Oxford University Press.

Dehaene, S., Piazza, M., Pinel. P., & Cohen L. (2003). Three parietal circuits for number processing. *Cogn. Neuropsychol*. 20, 487–506.

DeWolf, M., Chiang, J.N., Bassok, M., Holyoak, K.J & Monti, M.M. (2016). Neural Representations of Magnitude for Natural and Rational Numbers, *NeuroImage*, 141, 304-312. http://dx.doi.org/10.1016/j.neuroimage.2016.07.052

DeWolf, M., Grounds, M., Bassok, M. & Holyoak, K.J. (2014). Magnitude comparison with different types of rational numbers. *J. Exp. Psychol. Hum. Percept. Perform.* 40 (1), 71–82.

Empson, S. & Levi, L. (2011). Extending children's mathematics: Fractions and decimals: Innovations in cognitively guided instruction. Portsmouth, NH:Heinemann. pp. 178-216.

English, L., & Halford, G. (1995). Mathematics education: Models and processes. Mahwah, NJ: Lawrence Erlbaum. doi: 10.5860/choice.33-3963

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, 1(**2**), 7-26. Reprinted in Carpenter, T. P., Dossey, J. A., & Koehler, J. L. (Eds.) (2004), Classics in mathematics education research (pp. 48-58). Reston, VA: National Council of Teachers of Mathematics.

Foley, T., & Cawley, J. (2003). About the mathematics of division: Implications for students with learning disabilities. Exceptionality, 11, 131-150. doi: 10.1207/s15327035ex1103_02

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., . . . Changas, P. (2013). Improving At-Risk Learners' Understanding of Fractions. *Journal of Educational Psychology, 105*, 683-700. doi: 10.1037/a0032446

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., . . . Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude–treatment interaction. *Journal of Educational Psychology, 106*, 499-514. doi: 10.1037/a0034341

Gabriel, F., Coche, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions, *Frontiers in Psychology*, 4, 1-11. doi: 10.3389/fpsyg.2013.00715

Hannula, M. S. (2003). Locating fraction on a number line. In N. A. Paterman, B. J. Dougherty, & J. T Zilliox (Eds.), *Proceedings of the 27th conference of the International Group of the Psychology of Mathematics Education*, Vol. 3, 17-24. Honolulu, HI.

Hart, K, (2000). Mathematics Content and Learning Issues in the Middle Grades. In Mathematics Education in the Middle Grades: Teaching to Meet the Needs of Middle Grades Learners and to Maintain High Expectations, *Proceedings of a National Convocation and Action Conferences* (pp. 50-57). Washington, USA: National Academy Press.

Hecht, S. A., & Vagi, K. J. (2012). Patterns of strengths and weaknesses in children's knowledge about fractions. *Journal of Experimental Child Psychology, 111*, 212-229. doi: 10.1016/j.jecp.2011.08.012

Hiebert, J. (ed.). (1986). *Conceptual and Procedural Knowledge: the Case of Mathematics*. Hillsdale, NJ: Erlbaum.

Hurst, M. & Cordes, S. (2016). Rational-Number Comparison Across Notation: Fractions, Decimals and Whole Numbers, *Journal of Experimental Psychology: Human Perception and Performance*, 42(2), 281-293. http://dx.doi.org/10.1037/xhp0000140.

Ischebeck, A., Schocke, M., Delazer, M. (2009). The processing and representation of fractions within the brain: an fMRI investigation. *Neuroimage*. 47, 403–413. doi: 10.1016/j.neuroimage.2009.03.041. Epub 2009 Mar 25.

Jacob, S.N. & Nieder, A. (2009). Notation-independent representation of fractions in the human parietal cortex. *J. Neurosci.* 29, 4652–4657.

Jacob, S.N., Vallentin, D. & Nieder A. (2012) Relating magnitudes: the brain's code for proportions. *Trends Cogn. Sci.*, 16, 157–166. doi: 10.1016/j.tics.2012.02.002.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology, 116*, 45-58. doi: 10.1016/j.jecp.2013.02.001

Kamii, C. & Clark, F.B. (1995). Equivalent fractions: Their difficulty and educational implications. *Journal of Mathematical Behaviour*, 14 (4), 365–378.

Kerslake, D. (1986). Fractions: Children's Strategies and Errors. *A Report of the Strategies and Errorsin Secondary Mathematics Project*. England: NFER-NELSON Publishing Company Ltd.

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.). *Recent research on number learning*, (pp. 125–149). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Lamon, S.J.(2007). Rational number and proportional reasoning. In F.K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning*, pp.629-667. Charlotte, NC: Information Age Publishing.

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, 38, 201-221. doi:10.1016/j.dr.2015.07.008.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21(**1**), 16-32.

Mayer, R. E. (2010). Applying the science of learning to medical education. *Medical Education*, (44), 543–549. doi:10.1111/j.1365-2923.2010.03624.x.

Mayer, R. E. (2010). Seeking a science of instruction. *Instructional Science*, (38), 143–145. doi:10.1007/s11251-009-9113-x.

McLeod, R., & Newmarch, B. (2006). *Fractions, Math for Life*. National Research and Development Centre for Adult Literacy and Numeracy, London.

Moseley, B., & Okamoto, Y. (2008) Identifying fourth graders' understanding of rational number representations: A mixed methods approach. *School Science and Mathematics*. 108 (6), 238-250.

Moss, J. & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education* 30(2), 122-147.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: the origins and implications of whole number bias. *Educ. Psychologist* 40, 27–52. doi:10.1207/ s15326985ep4001_3

Opfer, J.E. & Siegler, R.S. (2012). Development of quantitative thinking. In Holyoak, K.J. & Morrison, R., (Eds). *Oxford Handbook of Thinking and Reasoning*, pp. 585–605, Oxford University Press, Cambridge.

Orpwood, G., Schollen, L., Leek, G., Marinelli-Henriques, P., & Assiri, H. (2012). College mathematics project 2011: Final report. Retrieved from http://collegemathproject.senecac.on.ca/cmp/en/pdf/FinalReport/2011/CMP _2011_Final_Report%20 -%2002Apr12%20pmh.pdf

Park, J., Flores, A., & Hohensee, C. (2016) Fraction as Numbers and Extensions of Numbers and Extensions of the Number System: Developing Activities Bases on Research. *Ohio Journal of Mathematics*. 73, 13-21.

Perle, M., Moran, R., & Lutkus, A. D. (2005). *NAEP 2004, trends in academic progress: Three decades of student performance in reading and mathematics*. Washington, DC: National Center for Education Statistics.

Post, T. R., Cramer, K. A., Lesh, R., Harel, G., & Behr, M. (1993). Curriculum Implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), Rational numbers: An integration of research (pp. 327-362). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: does one lead to the other? *J. Educ. Psychol*. 91, 175–189. doi: 10.1037/0022-0663.91.1.175

Schmithorst, V.J. & Brown, R.D. (2004). Empirical validation of the triple-code model of numerical processing for complex math operations using functional MRI and group Independent Component Analysis of the mental addition and subtraction of fractions. *Neuroimage*. 22, 1414–1420.

Schneider, M. and Siegler, R.S. (2010) Representations of the magnitudes of fractions. *J. Exp. Psychol. Hum. Percept. Perform*. 36, 1227–1238.

Schneider, M., & Stern, E. (2005). "Conceptual and procedural knowledge of a mathematics problem: their measurement and their causal interrelations," in *Paper Presented at the 27th Annual Meeting of the Cognitive Science Society (CSS)*, Stresa.

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*, 1994–2004. doi:10.1037/a0031200.

Siegler, R. S., Fazio, L.K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, 17, 13-19.

Siegler, R.S., Thompson, C.A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cogn. Psychol.* 62, 273–296. doi: 10.1016/j.cogpsych.2011.03.001

Stepanek, J., Leong, M., Griffin, L. & Lavelle, L. (2011). *Teaching by Design in Elementary Mathematics*. Corwin (In association with Education Northwest).

Stigler, J. W., Givvin, K. B., & Thompson, B. (2010). What community college developmental mathematics students understand about mathematics. *The MathAMATYC Educator*, *10*, 4–16.

Sweller, J. (1988). Cognitive Load During Problem Solving: Effects on Learning. *Cognitive Science,* 12 (2), 257–285.

Sweller, J., Van M. J. & Paas, F. (1998). Cognitive Architecture and Instructional Design. *Educational Psychology Review*, 10 (3), 251–296.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: a conceptual change approach, *Learn. Instr.* 14, 453–467. doi:10.1016/j. learninstruc.2004.06.013

Watanabe, T. (2002). Representations in Teaching and Learning Fractions. *Teaching Children Mathematics*, NCTM, 8(8), 457-463.

Wu, H. (2014). *Fractions, Decimals and Rational Numbers*, Report of the Learning Processes Task Group of the National mathematics Advisory Panel. Berkeley. doi: https://math.berkeley.edu/~wu/NMPfractions.pdf

Citation:
Pant, V. (2019). Conceptual understanding and procedural knowledge of fractions: How to learn and how to teach. In M. Shelley & S.A. Kiray (Ed.). *Education Research Highlights in Mathematics, Science and Technology 2019* (pp. 17-35). ISRES Publishing, ISBN: 978-605-69854-0-9.